DEFORMATION OF TENSOR PRODUCT (CO)ALGEBRAS VIA NON-(CO)NORMAL TWISTS

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ABSTRACT. We study new coalgebra structures on the tensor product of two coalgebras C and D by twisting the tensor product coalgebra via a twist map $\Psi : C \otimes D \to D \otimes C$. We deal with the general case in which the counit of the tensor product coalgebra is deformed as well. Some classes of such deformations are analyzed and a notion of equivalence of twists is discussed. We also present the dual deformation of tensor product algebras and provide examples.

1. INTRODUCTION

Twisted tensor product algebras $A \otimes_\Psi B$ arise as deformations of the tensor product algebra $(A \otimes B, (m_A \otimes m_B) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B))$ of two associative algebras A and B via linear maps $\Psi : B \otimes A \to A \otimes B$, with new multiplication $m_\Psi = (m_A \otimes m_B) \circ (\mathrm{id}_A \otimes \Psi \otimes \mathrm{id}_B)$. A compatibility condition (cf. (6.2)) between Ψ and the multiplication maps m_A, m_B provides a necessary and sufficient condition for the associativity of the twisted multiplication m_Ψ.

Dually a twisted tensor coalgebra $C \otimes_\Psi D$ of two coassociative coalgebras C, D is constructed on the vector space $C \otimes D$ in terms of a twisted coproduct $\Delta_\Psi = (\mathrm{id}_C \otimes \Psi \otimes \mathrm{id}_D) \circ (\Delta_C \otimes \Delta_D)$ out of a linear map $\Psi : C \otimes D \to D \otimes C$. For twists Ψ satisfying a suitable condition (cf. (2.2)), Δ_Ψ is coassociative.

Deformations of tensor products in this setting were first studied in [4] in the case of algebras. Nevertheless the idea of deforming the tensor product of two objects to obtain a new one is much older than that. Notable examples are the classical crossed products (see e.g. [8]), the quantum double of Drinfel'd [5] or the bicrossproduct and double cross product Hopf algebras of Majid [7]. In these constructions one deals with a much richer structure on and interplay between the two given objects which is then reflected on the resulting one. Disregarding this extra data, some of these concepts can be retrieved as particular instances of the theory of twisted tensor products.

Twisted tensor products can be studied in the framework of (co)algebra factorisations of a (co)algebra in two sub(co)algebras or in the opposite perspective of building a new object out of two given ones. Deformations of tensor products have been studied both in pure algebra and in connection to other branches of mathematics, notably non commutative geometry, and to physics. Some relevant spaces in noncommutative geometry and physics can be indeed recovered as particular twisted tensor products (see e.g. [6] and the references therein).

Historically, twisted products of (co)algebras have been studied for twisting maps Ψ which satisfy a (co)unital condition (cf. Def. 2.2, Def. 6.2) which ensures that the undeformed tensor (co)unit is still compatible with the deformed (co)product. In the present paper we address to the more general case of twists which are not necessarily

(co)normal. For different reasons we were most interested in the deformation of tensor product of coalgebras and so we present our results first in that case. Once we drop the requirement that the twist map Ψ is conormal, the new coalgebra $C \otimes_\Psi D$ might or might not admit a counit. In Sect. 3 we study twisted coalgebras associated to Z-conormal twists. This notion is a generalization of the former one and guarantees the existence of a compatible counit ε_Ψ for the twisted coproduct Δ_Ψ. These deformations still enjoy a universal property among factorized coalgebras (Thm 3.8), in analogy to the case of conormal twists treated in [3]. We also analyze a class of twisted coalgebras $C \otimes_{\Psi_\phi} D$ which are generated from twists which are intrinsically non conormal. These twists correspond to particular morphisms in the category $^D M^C$ of left D - right C-comodules and are in one-to-one correspondence with functionals ϕ on $C \otimes D$. The existence of a counit ε_{Ψ_ϕ} in this case can be expressed as a condition on ϕ. Although this class of twists consist of non-conormal ones, the resulting twisted coalgebras $(C \otimes_{\Psi_\phi} D)$ eventually turned out to be all isomorphic to the untwisted one $C \otimes D$ (albeit non trivially). Nevertheless, this class can be used to built new interesting deformations (see §4.2). We also briefly discuss a notion of equivalence of twists, but a cohomological interpretation of our deformations along the lines of [1, 4] is postponed. We conclude the paper by presenting the results in the dual case of the tensor product of two algebras.

The structure of the paper is as follows. In Sect. 2 we review some known results regarding twisted tensor product coalgebras relevant to our study. In particular we recall the factorization and universal properties for the twisted tensor coalgebra $C \otimes_\Psi D$ of two coalgebras by a conormal twist Ψ. In Sect. 3 we introduce the notion of Z-conormal twists and generalize in this framework the previously mentioned results. Sect. 4 describes the particular class of coassociative twisted tensor coalgebras $C \otimes_\Psi D$ which are generated from twists associated to functionals on $C \otimes D$. Further, in Sect. 4.2 we discuss of their use for the construction of new twisted tensor coalgebras out of old ones. In Sect. 5 we introduce a possible notion of equivalence of twists. Finally in Sect. 6 we present the dual results for the deformation via twists of the tensor product algebra $A \otimes B$ of two algebras.

2. Twisting tensor coalgebra co-structures

Notation. Coalgebras are over a commutative field k. Their coproduct and counit are as usual denoted with Δ, ε respectively. When it is necessary to avoid confusion, we specify to which coalgebra C they refer by writing Δ_C and ε_C. The unadorned tensor product $\otimes$ stands for tensor product over k, and τ is the flip map. We denote C^{cop} the coalgebra C endowed with the opposite coproduct $\Delta^{cop} := \tau \circ \Delta_C$. Given a tensor product of coalgebras $C \otimes D$ we write $\Delta_\otimes, \varepsilon_\otimes$ for the tensor product co-structures $\Delta_\otimes := (\mathrm{id}_C \otimes \tau \otimes \mathrm{id}_D)(\Delta_C \otimes \Delta_D)$ and $\varepsilon_\otimes := \varepsilon_C \otimes \varepsilon_D$. In case of Hopf algebras, the antipode is denoted with S. Summation over repeated indices is understood and we make use of Sweedler and Sweedler-like notations for coproduct and coactions.

Given C and D coassociative coalgebras, we are interested in defining a twisted coproduct on $C \otimes D$. Throughout the paper we call *twist map* a bilinear map $\Psi : C \otimes D \to D \otimes C$, $c \otimes d \mapsto d^{[\Psi]} \otimes c^{[\Psi]}$. Given a twist Ψ we also use the notation $\Psi' := \Psi\tau$.

The starting point of our investigation are the following results taken from [3, 4]. We follow the presentation of [3, §3].

Theorem 2.1. *Let Ψ be a twist map. The map $\Delta_\Psi : C \otimes D \to C \otimes D \otimes C \otimes D$ given by*

$$\Delta_\Psi := (\mathrm{id}_C \otimes \Psi \otimes \mathrm{id}_D) \circ (\Delta_C \otimes \Delta_D) \tag{2.1}$$

defines a coassociative coproduct on $C \otimes D$ if and only if the following diagram is commutative:

$$\begin{array}{ccc}
C \otimes D \otimes D & \xrightarrow{\Psi \otimes \mathrm{id}_D} D \otimes C \otimes D \xrightarrow{\mathrm{id}_D \otimes \Delta_C \otimes \mathrm{id}_D} D \otimes C \otimes C \otimes D & \qquad (2.2) \\
\end{array}$$

$$\begin{array}{ccc}
C \otimes D \otimes D & \xrightarrow{\Psi \otimes \mathrm{id}_D} D \otimes C \otimes D \xrightarrow{\mathrm{id}_D \otimes \Delta_C \otimes \mathrm{id}_D} D \otimes C \otimes C \otimes D \\
\ \uparrow{\scriptstyle \mathrm{id}_C \otimes \Delta_D} & & \downarrow{\scriptstyle \mathrm{id}_D \otimes \mathrm{id}_C \otimes \Psi} \\
C \otimes D & & D \otimes C \otimes D \otimes C \\
\ \downarrow{\scriptstyle \Delta_C \otimes \mathrm{id}_D} & & \uparrow{\scriptstyle \Psi \otimes \mathrm{id}_D \otimes \mathrm{id}_C} \\
C \otimes C \otimes D & \xrightarrow{\mathrm{id}_C \otimes \Psi} C \otimes D \otimes C \xrightarrow{\mathrm{id}_C \otimes \Delta_D \otimes \mathrm{id}_C} C \otimes D \otimes D \otimes C
\end{array}$$

that is

$$(\mathrm{id}_D \otimes \mathrm{id}_C \otimes \Psi)(\mathrm{id}_D \otimes \Delta_C \otimes \mathrm{id}_D)(\Psi \otimes \mathrm{id}_D)(\mathrm{id}_C \otimes \Delta_D) =$$
$$(\Psi \otimes \mathrm{id}_D \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \Delta_D \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \Psi)(\Delta_C \otimes \mathrm{id}_D) \quad (2.3)$$

We denote with $C \otimes_\Psi D$ the vector space $C \otimes D$ equipped with the twisted coproduct Δ_Ψ. The problem of defining a counit on $C \otimes D$ compatible with the twisted coproduct Δ_Ψ is usually solved by asking some further condition on Ψ. A possible choice is to restrict to twists which are compatible with the tensor counit $\varepsilon_\otimes = \varepsilon_C \otimes \varepsilon_D$.

Definition 2.2. *A twist map Ψ is said to be right conormal if*

$$(\varepsilon_D \otimes \mathrm{id}_C)\Psi = \mathrm{id}_C \otimes \varepsilon_D . \tag{2.4}$$

Similarly, it is said to be left conormal if

$$(\mathrm{id}_D \otimes \varepsilon_C)\Psi = \varepsilon_C \otimes \mathrm{id}_D . \tag{2.5}$$

It is said to be conormal when it is both right and left conormal.

Lemma 2.3. *Let Ψ be a twist map. The tensor counit $\varepsilon_\otimes = \varepsilon_C \otimes \varepsilon_D$ is compatible with the twisted coproduct Δ_Ψ, i.e. $(\mathrm{id} \otimes \varepsilon_\otimes)\Delta_\Psi = \mathrm{id}_{C \otimes D} = (\varepsilon_\otimes \otimes \mathrm{id})\Delta_\Psi$, if and only if Ψ is conormal.*

We notice that the coassociativity of Δ_Ψ and the property of $\varepsilon_\otimes$ to be a compatible counit are completely independent, in that there are examples of coassociative coproducts Δ_Ψ which do not admit $\varepsilon_\otimes$ as counit (or even more in general which do not admit counit at all) and there are conormal twists Ψ which do not satisfy the coassociativity condition (2.3).

The class which has been studied in the literature is the one of conormal twists. Under the assumption of conormality of Ψ the commutativity of the octagonal diagram (2.2) can be equivalently reduced to the commutativity of the following pentagonal diagrams, by further composing both members of (2.3) with respectively $(\mathrm{id}_D \otimes \varepsilon_C \otimes$

$\mathrm{id}_D \otimes \mathrm{id}_C$) and ($\mathrm{id}_D \otimes \mathrm{id}_C \otimes \varepsilon_D \otimes \mathrm{id}_C$):

$$
\begin{array}{ccc}
C \otimes D & \xrightarrow{\;\Psi\;} D \otimes C \xrightarrow{\;\Delta_D \otimes \mathrm{id}_C\;} & D \otimes D \otimes C \\
\scriptstyle{\mathrm{id}_C \otimes \Delta_D} \big\downarrow & & \big\uparrow \scriptstyle{\mathrm{id}_D \otimes \Psi} \\
C \otimes D \otimes D & \xrightarrow{\quad\;\Psi \otimes \mathrm{id}_D\quad\;} & D \otimes C \otimes D
\end{array}
\tag{2.6}
$$

which amounts to

$$(\mathrm{id}_D \otimes \Psi)(\Psi \otimes \mathrm{id}_D)(\mathrm{id}_C \otimes \Delta_D) = (\Delta_D \otimes \mathrm{id}_C)\Psi \tag{2.7}$$

and

$$
\begin{array}{ccc}
C \otimes D & \xrightarrow{\;\Psi\;} D \otimes C \xrightarrow{\;\mathrm{id}_D \otimes \Delta_C\;} & D \otimes C \otimes C \\
\scriptstyle{\Delta_C \otimes \mathrm{id}_D} \big\downarrow & & \big\uparrow \scriptstyle{\Psi \otimes \mathrm{id}_C} \\
C \otimes C \otimes D & \xrightarrow{\quad\;\mathrm{id}_C \otimes \Psi\quad\;} & C \otimes D \otimes C
\end{array}
\tag{2.8}
$$

which amounts to

$$(\Psi \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \Psi)(\Delta_C \otimes \mathrm{id}_D) = (id_D \otimes \Delta_C)\Psi \,. \tag{2.9}$$

Theorem 2.4. *Let C, D and Y be coalgebras. The following two conditions are equivalent:*

(1) *There exists a coalgebra isomorphism $Y \simeq C \otimes_\Psi D$ for some conormal twist map Ψ solution of (2.2);*

(2) *Y factorizes through C and D, i.e. there exist coalgebra morphisms $u_C : Y \to C$ and $u_D : Y \to D$ such that the map $\eta := (u_C \otimes u_D)\Delta_Y : Y \to C \otimes D$ is an isomorphism of vector spaces.*

The proof is based on the fact that if $C \otimes_\Psi D$ is a coalgebra associated to a conormal twist $\Psi : C \otimes D \to D \otimes C$, the maps

$$
\begin{aligned}
\pi_C &: C \otimes_\Psi D \to C, \quad c \otimes d \mapsto \varepsilon_D(d)\, c \\
\pi_D &: C \otimes_\Psi D \to D, \quad c \otimes d \mapsto \varepsilon_C(c)\, d
\end{aligned}
\tag{2.10}
$$

are coalgebra morphisms, and the map

$$\eta := (\pi_C \otimes \pi_D)\Delta_\Psi : C \otimes_\Psi D \to C \otimes D \tag{2.11}$$
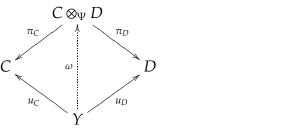
is an isomorphism of vector spaces. Indeed, it is $\eta(c \otimes d) = c \otimes d$. The opposite implication is constructive with the conormal twist map given by $\Psi = (u_D \otimes u_C) \circ \Delta_Y \circ \eta^{-1}$.

A twisted coalgebra $C \otimes_\Psi D$ enjoys the following universal property:

Proposition 2.5. *Let C and D be coalgebras, Ψ a conormal twist satisfying (2.2). Let Y be a coalgebra and $u_C : Y \to C$, $u_D : Y \to D$ coalgebra morphisms such that*

$$(u_D \otimes u_C) \circ \Delta_Y = \Psi \circ (u_C \otimes u_D) \circ \Delta_Y \tag{2.12}$$

4

then there exists a unique coalgebra morphism $\omega : Y \to C \otimes_\Psi D$ such that the following diagram commutes

$$
\begin{array}{ccc}
 & C \otimes_\Psi D & \\
\pi_C \nearrow & \uparrow \omega & \searrow \pi_D \\
C & & D \\
u_C \nwarrow & & \nearrow u_D \\
 & Y &
\end{array}
\tag{2.13}
$$

3. Non-conormal twists

We are interested in extending the study of solutions of the coassociativity condition (2.3) to the non-conormal case. We start by generalizing the notion of conormality.

Definition 3.1. *Let $Z : C \otimes D \to D \otimes C$, $c \otimes d \mapsto d^{[Z]} \otimes c^{[Z]}$ be a bilinear map. Set $\varepsilon^\tau_\otimes := \varepsilon_D \otimes \varepsilon_C$. A twist map Ψ is said to be right Z-conormal if it satisfies*

$$
\varepsilon((d^{[\Psi]})^{[Z]}) \varepsilon(c_{(1)}{}^{[Z]}) c_{(2)}{}^{[\Psi]} = \varepsilon(d) c
\tag{3.1}
$$

for all $c \in C$, $d \in D$. Similarly, it is said to be left Z-conormal if

$$
\varepsilon((c^{[\Psi]})^{[Z]}) \varepsilon(d_{(2)}{}^{[Z]}) d_{(1)}{}^{[\Psi]} = \varepsilon(c) d
\tag{3.2}
$$

for all $c \in C$, $d \in D$. It is said to be Z-conormal if it is both left and right Z-conormal.

Lemma 3.2. *Let Ψ be a twist map.*
  (i) *Ψ is left (resp. right) conormal if and only if Ψ is left (resp. right) τ-conormal.*
  (ii) *If Ψ is left (resp. right) conormal, then Ψ is left (resp. right) Ψ-conormal.*
  (iii) *If Ψ satisfies conditions (2.7)(2.9), then Ψ is left (resp. right) Ψ-conormal if and only if Ψ is left (resp. right) conormal.*

Proof. The first point is trivial. In the second, suppose Ψ is conormal. Then by applying twice (2.4) we get

$$
(\varepsilon_D \otimes \varepsilon_C \otimes \mathrm{id}_C)(Z \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \Psi)(\Delta_C \otimes \mathrm{id}_D) = (\varepsilon_C \otimes \varepsilon_D \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \Psi)(\Delta_C \otimes \mathrm{id}_D)
$$
$$
= (\varepsilon_C \otimes \varepsilon_D)(\Delta_C \otimes \mathrm{id}_D) = \mathrm{id}_C \otimes \varepsilon_D
$$

i.e. Ψ is right Ψ-conormal. Left Ψ-conormality is completely analogous. For the last point, suppose that Ψ satisfies (2.9). Then

$$
(\varepsilon_D \otimes \varepsilon_C \otimes \mathrm{id}_C)(\Psi \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \Psi)(\Delta_C \otimes \mathrm{id}_D) = (\varepsilon_D \otimes \varepsilon_C \otimes \mathrm{id}_C)(\mathrm{id}_D \otimes \Delta_C)\Psi
$$
$$
= (\varepsilon_D \otimes \mathrm{id}_C)\Psi
$$

so that right Z-conormality implies right conormality. For the left-version we use (2.7). $\square$

Lemma 3.3. *Consider $Z, Z' : C \otimes D \to D \otimes C$. Then the following two conditions are equivalent:*
  (i) *a twist map Ψ is Z conormal if and only if it is Z' conormal*
  (ii) *$\mathrm{Im}(Z - Z') \in \ker(\varepsilon^\tau_\otimes)$*

5

Proposition 3.4. *A twist map Ψ which satisfies* (2.3) *is Z-conormal for some Z if and only if*

$$\varepsilon_Z := \varepsilon_\otimes^\tau \circ Z = (\varepsilon_D \otimes \varepsilon_C) \circ Z \tag{3.3}$$

defines a counit for the twisted coproduct Δ_Ψ.

Proof. Assume Ψ is Z-conormal, so in particular it is left Z-conormal. Then

$$\begin{aligned}
(\mathrm{id}_{C\otimes D} \otimes \varepsilon_Z)\Delta_\psi &= (\mathrm{id}_{C\otimes D} \otimes \varepsilon_Z)(\mathrm{id}_C \otimes \Psi \otimes \mathrm{id}_D)(\Delta_C \otimes \Delta_D) \\
&= \Big(\mathrm{id}_C \otimes \big((\mathrm{id}_D \otimes \varepsilon_Z)(\Psi \otimes \mathrm{id}_D)(\mathrm{id}_C \otimes \Delta_D)\big)\Big)(\Delta_C \otimes \mathrm{id}_D) \\
&= (\mathrm{id}_C \otimes \varepsilon_C \otimes \mathrm{id}_D)(\Delta_C \otimes \mathrm{id}_D) = \mathrm{id}_{C\otimes D}\,.
\end{aligned}$$

Similarly, by using the right Z-conormality we prove $(\varepsilon_Z \otimes \mathrm{id}_{C\otimes D})\Delta_\Psi = \mathrm{id}_{C\otimes D}$. On the other side, suppose ε_Z is a counit for Δ_Ψ. Then $(\mathrm{id}_{C\otimes D} \otimes \varepsilon_Z)\Delta_\psi = \mathrm{id}_{C\otimes D}$ and by applying $\varepsilon_C \otimes \mathrm{id}_D$ to both sides we get the left Z-conormality condition (3.2). By applying $\mathrm{id}_C \otimes \varepsilon_D$ we get the right Z conormality condition (3.1). $\qquad\square$

In the following, when not necessary to specify the map Z, we will use the notation ε_Ψ to indicate the counit of a given Z-conormal twist Ψ. Note that it is not restrictive to assume a counit of the form $\varepsilon_Z = \varepsilon_\otimes^\tau \circ Z$. Indeed, given a generic counit ε_Ψ, we can always find a Z such that $\varepsilon_\Psi = \varepsilon_Z$, for example $Z(c \otimes d) = \varepsilon_\Psi(c_{(1)} \otimes d_{(2)})\, d_{(1)} \otimes c_{(2)}$.

3.1. **Universal properties and factorization.** In this section, if not otherwise stated, we consider generic twist maps Ψ such that the (non necessarily coassociative) coproduct Δ_Ψ admits compatible counit ε_Ψ. With an abuse of terminology sometimes we refer to ε_Ψ as a compatible counit for Ψ, rather than for Δ_Ψ. Further properties, for example that Ψ satisfies the coassociativity condition (2.3), are explicitly mentioned when relevant.

Consider the projection maps $p_C : C \otimes_\Psi D \to C$, $p_D : C \otimes_\Psi D \to D$ defined respectively by

$$\begin{aligned}
p_C &:= (\mathrm{id}_C \otimes \varepsilon_\Psi)(\Delta_C \otimes \mathrm{id}_D)\,, \quad p_C(c \otimes d) = c_{(1)}\varepsilon_\Psi(c_{(2)} \otimes d)\,; \\
p_D &:= (\varepsilon_\Psi \otimes \mathrm{id}_D)(\mathrm{id}_C \otimes \Delta_D)\,, \quad p_D(c \otimes d) = \varepsilon_\Psi(c \otimes d_{(1)})\, d_{(2)}\,.
\end{aligned} \tag{3.4}$$

Lemma 3.5. *The projection maps p_C and p_D are coalgebra morphisms.*

Proof. We have to show that $\varepsilon_C \circ p_C = \varepsilon_\Psi$ and $(p_C \otimes p_C) \circ \Delta_\Psi = \Delta_C \circ p_C$. The first statement is trivial. For the second one we have

$$\begin{aligned}
(p_C \otimes p_C) \circ \Delta_\Psi(c \otimes d) &= (p_C \otimes p_C)(c_{(1)} \otimes (d_{(1)})^{[\Psi]} \otimes (c_{(2)})^{[\Psi]} \otimes d_{(2)}) \\
&= (\mathrm{id}_C \otimes p_C)(c_{(1)} \otimes \varepsilon_\Psi(c_{(2)} \otimes (d_{(1)})^{[\Psi]}) \otimes (c_{(3)})^{[\Psi]} \otimes d_{(2)}) \\
&= (\mathrm{id}_C \otimes p_C)(c_{(1)} \otimes c_{(2)} \otimes d) \\
&= c_{(1)} \otimes c_{(2)}\varepsilon_\Psi(c_{(3)} \otimes d) \\
&= \Delta_C \circ p_C(c \otimes d)
\end{aligned}$$

The proof for p_D is completely analogous. $\qquad\square$

These maps generalize the projections π_C, π_D (2.10) to generic twists; indeed for Ψ conormal we recover $p_C = \pi_C$ and $p_D = \pi_D$.

Let $(C \otimes D)'$ be the set of linear functionals $\phi : C \otimes D \to k$. We consider two different algebra structures on $(C \otimes D)'$; given $\phi, \phi' \in (C \otimes D)'$ we can multiply them via the convolution product

$$(\phi * \phi')(c \otimes d) = \phi(c_{(1)} \otimes d_{(1)}) \, \phi'(c_{(2)} \otimes d_{(2)}) \tag{3.5}$$

or via a $\star$-product defined as

$$(\phi \star \phi')(c \otimes d) = \phi(c_{(2)} \otimes d_{(1)}) \, \phi'(c_{(1)} \otimes d_{(2)}) . \tag{3.6}$$

Both algebras $((C \otimes D)', *)$ and $((C \otimes D)', \star)$ are associative, with unit element the tensor counit $\varepsilon_\otimes$. Indeed $((C \otimes D)', \star) = ((C^{cop} \otimes D)', *)$, i.e. the $\star$ product is just the convolution product on $C^{cop} \otimes D'$. Then if C is co-commutative it is $((C \otimes D)', \star) = ((C \otimes D)', *)$, while if D is co-commutative $(C^{cop} \otimes D)^{cop} = (C \otimes D^{cop}) = (C \otimes D)$ implies that $((C \otimes D)', \star) = ((C \otimes D)', *^{op})$.

Proposition 3.6. *Let Ψ be a twist with compatible counit ε_Ψ. Let p_C, p_D be the coalgebra maps defined in (3.4). The counit ε_Ψ is invertible in $[(C \otimes D)', \star]$ if and only if the map $\mu : C \otimes_\Psi D \to C \otimes D$,*

$$\mu(c \otimes d) := (p_C \otimes p_D)\Delta_\Psi(c \otimes d) = \varepsilon_\Psi(c_{(2)} \otimes d_{(1)}) \, c_{(1)} \otimes d_{(2)} ,$$

is an isomorphism of vector spaces.

Proof. We first assume that ε_Ψ is invertible in $[(C \otimes D)', \star]$. Denote $\varepsilon_\Psi^\star$ the inverse of ε_Ψ with respect to the product $\star$. We claim that

$$\mu^{-1}(c \otimes d) = \varepsilon_\Psi^\star(c_{(2)} \otimes d_{(1)}) \, c_{(1)} \otimes d_{(2)} .$$

By a direct computation

$$\begin{aligned}
\mu^{-1} \circ \mu(c \otimes d) &= \mu^{-1}(\varepsilon_\Psi(c_{(2)} \otimes d_{(1)})c_{(1)} \otimes d_{(2)}) \\
&= \varepsilon_\Psi(c_{(3)} \otimes d_{(1)}\varepsilon_\Psi^\star(c_{(2)}d_{(2)})c_{(1)} \otimes d_{(3)} \\
&= (\varepsilon_\Psi \star \varepsilon_\Psi^\star)(c_{(2)} \otimes d_{(1)})c_{(1)} \otimes d_{(2)} = c \otimes d ; \\
\mu \circ \mu^{-1}(c \otimes d) &= \mu(\varepsilon_\Psi^\star(c_{(2)} \otimes d_{(1)})c_{(1)} \otimes d_{(2)}) \\
&= \varepsilon_\Psi^\star(c_{(3)} \otimes d_{(1)}\varepsilon_\Psi(c_{(2)} \otimes d_{(2)}))c_{(1)} \otimes d_{(3)} \\
&= (\varepsilon_\Psi^\star \star \varepsilon_\Psi)(c_{(2)} \otimes d_{(1)})c_{(1)} \otimes d_{(2)} = c \otimes d .
\end{aligned}$$

For the opposite implication, suppose μ is invertible. Then we show that ε_Ψ is invertible with inverse $\varepsilon_\Psi^\star := \varepsilon_\otimes \circ \mu^{-1}$. On the one hand

$$(\varepsilon_\Psi \star \varepsilon_\otimes \circ \mu^{-1})(c \otimes d) = (\varepsilon_\otimes \circ \mu^{-1}) \left(\varepsilon_\Psi(c_{(2)} \otimes d_{(1)}) \, c_{(1)} \otimes d_{(2)} \right) = (\varepsilon_\otimes \circ \mu^{-1})(\mu(c \otimes d)) = \varepsilon_\otimes(c \otimes d) ,$$

so that $\varepsilon_\Psi^\star$ is the right inverse of ε_Ψ. Using this result we can prove that $\alpha(c \otimes d) := \varepsilon_\Psi^\star(c_{(2)} \otimes d_{(1)}) \, c_{(1)} \otimes d_{(2)}$ coincides with μ^{-1}. Indeed

$$\begin{aligned}
(\alpha \circ \mu)(c \otimes d) &= \varepsilon_\Psi(c_{(2)} \otimes d_{(1)})\alpha(c_{(1)} \otimes d_{(2)}) = \varepsilon_\Psi(c_{(3)} \otimes d_{(1)})\varepsilon_\Psi^\star(c_{(2)} \otimes d_{(2)}) \, c_{(1)} \otimes d_{(3)} \\
&= (\varepsilon_\Psi \star \varepsilon_\Psi^\star)(c_{(2)} \otimes d_{(1)})c_{(1)} \otimes d_{(2)} = c \otimes d
\end{aligned}$$

hence α is the left inverse of μ, and therefore $\alpha = \mu^{-1}$ being μ invertible. We can now prove that $\varepsilon_\Psi^\star$ is the left inverse of ε_Ψ:

$$(\varepsilon_\Psi^\star \star \varepsilon_\Psi)(c \otimes d) = \varepsilon_\Psi \left(c_{(1)} \otimes d_{(2)}\varepsilon_\Psi^\star(c_{(2)} \otimes d_{(1)}) \right) = \varepsilon_\Psi(\mu^{-1}(c \otimes d)) = \varepsilon_\otimes(c \otimes d)$$

since $\varepsilon_\Psi = \varepsilon_\otimes \circ \mu$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Note that the above result holds for generic twists, not necessarily solutions of (2.3).
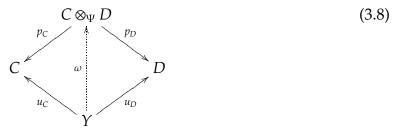
As a consequence of Thm. 2.4 we have the following

Corollary 3.7. *Let $(C \otimes_\Psi D, \Delta_\Psi, \varepsilon_\Psi)$ be a twisted coalgebra associated to a twist map Ψ which satisfies the coassociativity condition (2.3). Suppose there exist two coalgebra morphisms $p_C : C \otimes_\Psi D \to C$, $p_D : C \otimes_\Psi D \to D$ such that the map $\mu = (p_C \otimes p_D)\Delta_\Psi$ is invertible. Then $\widetilde{\Psi} := (p_D \otimes p_C)\Delta_\Psi \mu^{-1}$ is a conormal twist, and $\mu : C \otimes_\Psi D \to C \otimes_{\widetilde{\Psi}} D$ realizes a coalgebra isomorphism.*

We have the following universal characterization of the twisted coalgebra $C \otimes_\Psi D$ (cf. Prop. 2.5).

Theorem 3.8. *Let $C \otimes_\Psi D = (C \otimes D, \Delta_\Psi, \varepsilon_\Psi)$ be a twisted coalgebra for some twist Ψ solution of (2.3), and p_C, p_D the projections introduced in (3.4). Let Y be a coassociative coalgebra, together with coalgebra morphisms $u_C : Y \to C$, $u_D : Y \to D$ such that*

$$(u_D \otimes u_C) \circ \Delta_Y = \Psi \circ (u_C \otimes u_D) \circ \Delta_Y, \qquad \varepsilon_\Psi (u_C \otimes u_D)\Delta_Y = \varepsilon_Y. \qquad (3.7)$$

Then there exists a coalgebra morphism $\omega : Y \to C \otimes_\Psi D$ such that $u_C = p_C \circ \omega$ and $u_D = p_D \circ \omega$, i.e. the following diagram commutes

$$\begin{array}{ccc} & C \otimes_\Psi D & \qquad (3.8) \\ \end{array}$$

Proof. Set $\omega := (u_C \otimes u_D) \circ \Delta_Y$. The proof that ω is a coalgebra map is the same of Corollary 3.7 for the map μ: for $y \in Y$ we have

$$\begin{aligned} (\Delta_\Psi \circ \omega)(y) &= \Delta_\Psi(u_C(y_{(1)}) \otimes u_D(y_{(2)})) \\ &= (1 \otimes \Psi \otimes 1) \circ (\Delta_C \otimes \Delta_D)(u_C(y_{(1)}) \otimes u_D(y_{(2)})) \\ &= (1 \otimes \Psi \otimes 1)(u_C(y_{(1)}) \otimes u_C(y_{(2)}) \otimes u_D(y_{(3)}) \otimes u_D(y_{(4)})) \\ &= u_C(y_{(1)}) \otimes \Psi\big((u_C \otimes u_D) \circ \Delta_Y(y_{(2)}) \big) \otimes u_D(y_{(3)}) \\ &= u_C(y_{(1)}) \otimes (u_D \otimes u_C) \circ \Delta_Y(y_{(2)}) \otimes u_D(y_{(3)}) \\ &= u_C(y_{(1)}) \otimes u_D(y_{(2)}) \otimes u_C(y_{(3)}) \otimes u_D(y_{(4)}) \\ &= (\omega \otimes \omega) \circ \Delta_Y(y) . \end{aligned}$$

The second condition in (3.7) now reads $\varepsilon_\Psi \circ \omega = \varepsilon_Y$, which is the intertwining property for the counit. We finally show that ω indeed makes the diagram (3.8) commutative.

We compute $p_C \circ \omega(y)$ for a generic $y \in Y$, using the fact that ω is a coalgebra map:

$$\begin{aligned}
p_C \circ \omega\,(y) &= p_C \circ (u_C \otimes u_D) \circ \Delta_Y(y) \\
&= (1 \otimes \tilde{\varepsilon}) \circ (\Delta_C \otimes 1)(\,u_C(y_{(1)}) \otimes u_D(y_{(2)})\,) \\
&= (1 \otimes \tilde{\varepsilon})(\,u_C(y_{(1)}) \otimes u_C(y_{(2)}) \otimes u_D(y_{(3)})\,) \\
&= u_C(y_{(1)})\,\tilde{\varepsilon}(\,\omega(y_{(2)})\,) \\
&= u_C(y_{(1)})\,\varepsilon_Y(y_{(2)}) = u_C(y)\,.
\end{aligned}$$

The computation for $u_D = p_D \circ \omega(y)$ is similar. $\qquad\qquad\qquad\qquad\square$

Remark 3.9. In the above Theorem the coalgebra map ω is unique for twists Ψ such that $\mu = (p_C \otimes p_D)\Delta_\Psi$ is injective. Indeed suppose there exist two coalgebra maps ω, ω' which make the diagram (3.8) commute. Then consider the coalgebra map $\Omega = \omega - \omega'$; its image is a sub-coalgebra in $C \otimes_\Psi D$, on which both p_C and p_D vanish. Hence $\mu(x) = 0$ for every $x \in im(\Omega)$, so it must be $\Omega = 0$.

Remark 3.10. Suppose Y is itself a twisted coalgebra, $Y = C \otimes_\chi D$ for some twist χ solution of (2.3) admitting counit, and the map $\mu_\chi = (u_C \otimes u_D)\Delta_Y$ is invertible (i.e. the case of Corollary 3.7). Then condition (3.7) is satisfied by the conormal twist $\Psi = \widetilde{\chi} = (u_D \otimes u_C)\Delta_Y \mu_\chi^{-1}$ and the coalgebra isomorphism $\omega : C \otimes_{\widetilde{\chi}} D \to C \otimes_\chi D$ is the same of Corollary 3.7.

4. Twists from functionals

In this section we describe a class of solutions of condition (2.3), which are intrinsically non conormal. In particular this shows that once we drop the request of conormality, conditions (2.7)(2.9) are no longer necessary for the coassociativity of Δ_Ψ.

The basic idea is to solve (2.3) by moving the map Ψ to the left in both members; this is possible if we ask Ψ to intertwine suitable coactions of C and D. We formalize this property in a categorical language as follows.

We denote with $^D M^C$ the category of left D - right C comodules which have commuting left and right coactions; we denote by $Obj(^D M^C)$ its objects and by $Mor(^D M^C)$ its morphisms. We remember that as long as C, D are only coalgebras and not bialgebras, $^D M^C$ does not have a monoidal structure.

We observe that the map

$$_D\Phi = (\Delta_D \otimes 1) : D \otimes C \to D \otimes D \otimes C$$

defines a left coaction of D on $D \otimes C$, and that similarly, the map

$$\Phi_C = (1 \otimes \Delta_C) : D \otimes C \to D \otimes C \otimes C$$

defines a right C coaction of C on $D \otimes C$. These coactions $_D\Phi$ and Φ_C commute. Hence $D \otimes C \in Obj(^D M^C)$. The coalgebra $C \otimes D$ is an object in $^D M^C$ too:

$$_D\Phi^\tau = (1 \otimes \tau) \circ \Phi_D \circ \tau = (1 \otimes \tau) \circ (\Delta_D \otimes 1) \circ \tau : C \otimes D \to D \otimes C \otimes D$$

defines a left coaction of D on $C \otimes D$ and the map

$$\Phi_C^\tau = (\tau \otimes 1) \circ \Phi_C \circ \tau = (\tau \circ 1) \circ (1 \otimes \Delta_C) \circ \tau : C \otimes D \to C \otimes D \otimes C$$

defines a right C coaction of C on $D \otimes C$. The two coactions $_D\Phi^\tau$ and Φ_C^τ commute.

Theorem 4.1. *Any twist map* $\Psi \in Mor(^D M^C)$ *is a solution of* (2.3), *hence it defines a coassociative coproduct* Δ_Ψ *on* $C \otimes D$.

Proof. In the right-hand side of (2.3) we can use the fact that Ψ is a morphism of left D-comodules, i.e. $_D\Phi \circ \Psi = (id_D \otimes \Psi) \circ {_D\Phi^\tau}$, to write

$$(id_C \otimes \Delta_D \otimes id_C)(id_C \otimes \Psi) = (id_C \otimes {_D\Phi})(id_C \otimes \Psi) = (id_C \otimes id_D \otimes \Psi)(id_C \otimes {_D\Phi^\tau}).$$

Similarly in the left-hand side we can use the hypothesis that Ψ is a morphism of right C-comodules, i.e. $\Phi_C \circ \Psi = (\Psi \otimes id_C) \circ \Phi_C^\tau$, to write

$$(id_D \otimes \Delta_C \otimes id_D)(\Psi \otimes id_D) = (\Phi_C \otimes id_D)(\Psi \otimes id_D) = (\Psi \otimes id_C \otimes id_D)(\Phi_C^\tau \otimes id_D).$$

Hence (2.3) is satisfied provided

$$(\Psi \otimes \Psi) \circ (\Phi_C^\tau \otimes id_D) \circ (id_C \otimes \Delta_D) = (\Psi \otimes \Psi) \circ (id_C \otimes {_D\Phi^\tau}) \circ (\Delta_c \otimes id_D)$$

and in fact this equation holds because

$$(\Phi_C^\tau \otimes id_D)(id_C \otimes \Delta_D) = (id_C \otimes {_D\Phi^\tau})(\Delta_c \otimes id_D) = \Delta_\otimes$$

as one can easily verify on a generic element $c \otimes d \in C \otimes D$. $\qquad\square$

It is useful to notice that the property $\Psi \in Mor(^D M^C)$ can be equivalently expressed in terms of the map $\Psi' = \Psi \circ \tau : D \otimes C \to D \otimes C$ that will be also referred to as a twist in the following. It turns out that Ψ is a morphism in $^D M^C$ if and only if Ψ' does, that is if and only if

$$(\Delta_D \otimes id_C)\Psi' = (id_D \otimes \Psi')(\Delta_D \otimes id_C) \tag{4.1}$$

$$(id_D \otimes \Delta_C)\Psi' = (\Psi' \otimes id_C)(id_D \otimes \Delta_C). \tag{4.2}$$

This allows to conclude easily that

Corollary 4.2. *The space of solutions of* (4.1) *and* (4.2) *is a unital algebra with multiplication given by the composition of morphisms.*

We observe that conversely, it is not clear (at least to us) whether the composition of two twists which are solutions of (2.2) still gives a solution of the same equation. The same question remains open for the solutions of (2.6) and (2.8).

Furthermore, we get a simplified version of the above conditions (4.1) and (4.2) by applying respectively $id_D \otimes \varepsilon_D \otimes id_C$ and $id_D \otimes \varepsilon_C \otimes id_C$ to them. On the generic element $d \otimes c \in D \otimes C$ we have

$$d^{[\Psi]} \otimes c^{[\Psi]} = d_{(1)} \varepsilon_D\big((d_{(2)})^{[\Psi]}\big) \otimes c^{[\Psi]} \tag{4.3}$$

$$d^{[\Psi]} \otimes c^{[\Psi]} = d^{[\Psi]} \otimes \varepsilon_C\big((c_{(1)})^{[\Psi]}\big) c_{(2)}. \tag{4.4}$$

The previous identities are indeed equivalent to (4.1) and (4.2) (to prove the opposite implication it is enough to apply $\Delta_C \otimes id$, resp $id \otimes \Delta_D$ to (4.3), resp (4.4)) and can be used to simplify the expression of the twisted coproduct as

$$\Delta_\Psi(c \otimes d) = c_{(1)} \otimes (d_{(1)})^{[\Psi]} \otimes (c_{(2)})^{[\Psi]} \otimes d_{(2)}$$
$$= \varepsilon_C\big((c_{(2)})^{[\Psi]}\big)\, \varepsilon_D\big((d_{(2)})^{[\Psi]}\big)\, c_{(1)} \otimes d_{(1)} \otimes c_{(3)} \otimes d_{(3)}. \tag{4.5}$$

This shows that the coproduct is only 'mildly twisted'. Nevertheless, apart from the trivial case $\Psi = \tau$, this deformation turns out to be intrinsically non-conormal, as the next Lemma shows.

Lemma 4.3. *Consider a twist $\Psi \in Mor(^{D}M^{C})$. Then Ψ is conormal if and only if Ψ is the flip map τ.*

Proof. Suppose that Ψ is conormal. Then by using (4.3) we have

$$\Psi(c \otimes d) = d^{[\Psi]} \otimes c^{[\Psi]} = d_{(1)} \otimes \varepsilon_D\left((d_{(2)})^{[\Psi]}\right) \otimes c^{[\Psi]} = d_{(1)}\varepsilon_D(d_{(2)}) \otimes c = d \otimes c\,.$$

The opposite implication is trivial. $\qquad\square$

Remark 4.4. The possibility to compose twists $\Psi' : D \otimes C \to D \otimes C$, $\Psi' \in Mor(^{D}M^{C})$, could raise the following question about the need to consider non-conormal twists (and therefore deformed counits). Given a twisted coalgebra $(C \otimes_{\Psi} D, \Delta_{\Psi}, \varepsilon_{\Psi})$, one might wonder whether it is possible to find a second twist such that the composition of the two is conormal (i.e. we restore $\varepsilon_{\otimes}$ as the compatible counit). A positive answer would implies that we can always, up to some 'gauge equivalence' (the second twist), consider conormal twists without loss of generality. The previous Lemma shows that this is not the case for the class of twists we are considering. Since the only conormal twist in $Mor(^{D}M^{C})$ is the trivial one $\Psi = \tau$, the only way to 'untwist' the counit is by composing it with the inverse twist, thus 'untwisting' the coproduct as well.

We proceed now with the study of this class of solutions of (2.2) by showing that there is a one to one correspondence between the twists which are solutions of (4.1)(4.2) and the functionals on $C \otimes D$. This provides an easier description of the resulting twisted coalgebra, expressing several properties (existence of counit, composition of twists, universal properties etc) in terms of the functionals themselves.
In the remaining of the section we will denote by Tw the set of twist maps $\Psi' : D \otimes C \to D \otimes C$ such that $\Psi' \in Mor(^{D}M^{C})$.
As already observed before (Corollary 4.2) the set Tw becomes an algebra $(\mathsf{Tw}, \circ)$ with the composition $\circ$ of morphisms in $Mor(^{D}M^{C})$.

Theorem 4.5. *There is an algebra isomorphism $F : (\mathsf{Tw}, \circ) \to \left((C \otimes D)', \star\right)$ sending a twist map Ψ' into the associated functional*

$$F(\Psi') = \phi_{\Psi} := \varepsilon_{\otimes}^{\tau} \circ \Psi' \circ \tau = (\varepsilon_D \otimes \varepsilon_C) \circ \Psi\,.$$

Proof. We first show that F is invertible. Given $\phi \in (C \otimes D)'$, set for all $c \in C$, $d \in D$

$$(F^{-1}(\phi))(d \otimes c) = \Psi'_{\phi}(d \otimes c) := \phi(c_{(1)} \otimes d_{(2)})\, d_{(1)} \otimes c_{(2)}\,.$$

The map Ψ'_{ϕ} satisfies the conditions (4.1)(4.2), namely $\Psi'_{\phi} \in \mathsf{Tw}$; indeed

$$(\Delta_D \otimes id_C)\Psi'_{\phi}(d \otimes c) = \phi(c_{(1)} \otimes d_{(2)})\,(\Delta_D \otimes id_C)(d_{(1)} \otimes c_{(2)}) = \phi(c_{(1)} \otimes d_{(3)})\, d_{(1)} \otimes d_{(2)} \otimes c_{(2)}$$

and

$$(id_D \otimes \Psi'_{\phi})(\Delta_D \otimes id_C)(d \otimes c) = (id_D \otimes \Psi'_{\phi})(d_{(1)} \otimes d_{(2)} \otimes c) = \phi(c_{(1)} \otimes d_{(3)})\, d_{(1)} \otimes d_{(2)} \otimes c_{(2)}\,.$$

Similarly

$$(\mathrm{id}_D \otimes \Delta_C)\Psi'_\phi(d \otimes c) = \phi(c_{(1)} \otimes d_{(2)})\,(\mathrm{id}_D \otimes \Delta_C)(d_{(1)} \otimes c_{(2)}) = \phi(c_{(1)} \otimes d_{(2)})\,d_{(1)} \otimes c_{(2)} \otimes c_{(3)}$$

and

$$(\Psi'_\phi \otimes \mathrm{id}_C)(\mathrm{id}_D \otimes \Delta_C)(d \otimes c) = (\Psi'_\phi \otimes \mathrm{id}_C)(d \otimes c_{(1)} \otimes c_{(2)}) = \phi(c_{(1)} \otimes d_{(2)})\,d_{(1)} \otimes c_{(2)} \otimes c_{(3)}\,.$$

We verify that $FF^{-1} = F^{-1}F = \mathrm{id}$:

$$\phi_{(\Psi_\phi)}(c \otimes d) = \varepsilon^\tau_\otimes(\Psi_\phi)(c \otimes d)) = \phi(c_{(1)} \otimes d_{(2)})\,\varepsilon_D(d_{(1)})\,\varepsilon_C(c_{(2)}) = \phi(c \otimes d)$$

and similarly

$$\Psi'_{(\phi_\Psi)}(d \otimes c) = \phi_\Psi(c_{(1)} \otimes d_{(2)})\,d_{(1)} \otimes c_{(2)} = \varepsilon_D((d_{(2)})^{[\Psi]})\varepsilon_C((c_{(1)})^{[\Psi]})\,d_{(1)} \otimes c_{(2)} = \Psi'(d \otimes c)$$

where in the last equality we used (4.3) and (4.4). The maps F and F^{-1} are linear. The compatibility with the algebra structure follows from $F(\mathrm{id}_{D \otimes C}) = \phi_{(\mathrm{id}_{D \otimes C})} = \varepsilon_\otimes$ and

$$\phi_{(\Psi_2 \circ \tau \circ \Psi_1)}(c \otimes d) = \varepsilon^\tau_\otimes\big((d^{[\Psi_1]})^{[\Psi_2]} \otimes (c^{[\Psi_1]})^{[\Psi_2]}\big)$$

$$(\phi_{\Psi_2} \star \phi_{\Psi_1})(c \otimes d) = \phi_{\Psi_2}(c_{(2)} \otimes d_{(1)})\,\phi_{\Psi_1}(c_{(1)} \otimes d_{(2)})$$

$$= \varepsilon^\tau_\otimes\big((d_{(1)})^{[\Psi_2]} \otimes (c_{(2)})^{[\Psi_2]}\big)\,\varepsilon^\tau_\otimes\big((d_{(2)})^{[\Psi_1]} \otimes (c_{(1)})^{[\Psi_1]}\big)$$

$$= (\varepsilon^\tau_\otimes \circ \Psi'_2)\big(\varepsilon_D((d_{(2)})^{[\Psi_1]})\,d_{(1)} \otimes \varepsilon_C((c_{(1)})^{[\Psi_1]})\,c_{(2)}\big)$$

$$= (\varepsilon^\tau_\otimes \circ \Psi'_2)(d^{[\Psi_1]} \otimes c^{[\Psi_1]}) = \varepsilon^\tau_\otimes\big((d^{[\Psi_1]})^{[\Psi_2]} \otimes (c^{[\Psi_1]})^{[\Psi_2]}\big)\,.$$

$\square$

The previous result is a particular instance of the Hom-tensor relations in $^D M^C \simeq {}^{D \otimes C^{cop}} M$. For any k-coalgebra P, $N \in {}^P M$, V a k-vector space, there exists an isomorphism of k-vector spaces (see e.g. [2, §3.10]):

$$Mor(N, P \otimes V) \xrightarrow{\simeq} Hom_k(N, V)$$

which becomes an algebra isomorphism for $N = P$ and $V = k$.

The expression of the twisted coproduct Δ_Ψ in (4.5) can now be rewritten as

$$\Delta_\Psi(c \otimes d) = \phi_\Psi(c_{(2)} \otimes d_{(2)})\,c_{(1)} \otimes d_{(1)} \otimes c_{(3)} \otimes d_{(3)}\,. \tag{4.6}$$

In the following, whenever we want to emphasize the role of the functional ϕ, we write Δ_ϕ for the twisted coproduct Δ_{Ψ_ϕ}.

Proposition 4.6. *Let $\Psi' \in \mathsf{Tw}$. The twisted coproduct Δ_Ψ admits a counit if and only if ϕ_Ψ is invertible with respect to the convolution product, and in that case ϕ_Ψ^{-1} is the twisted counit.*

Proof. Suppose Δ_Ψ admits counit ε_Ψ. Then from $(\varepsilon_\Psi \otimes \mathrm{id}_{C \otimes D})\Delta_\Psi = (\mathrm{id}_{C \otimes D} \otimes \varepsilon_\Psi)\Delta_\Psi = \mathrm{id}_{C \otimes D}$ we get respectively

$$(\varepsilon_\Psi \otimes \mathrm{id}_{C \otimes D})\Delta_\Psi(c \otimes d) = \varepsilon_\Psi(c_{(1)} \otimes d_{(1)})\,\phi_\Psi(c_{(2)} \otimes d_{(2)})\,c_{(3)} \otimes d_{(3)}$$

$$= (\varepsilon_\Psi * \phi_\Psi)(c_{(1)} \otimes d_{(1)})\,c_{(2)} \otimes d_{(2)}\,,$$

$$(\mathrm{id}_{C \otimes D} \otimes \varepsilon_\Psi)\Delta_\Psi(c \otimes d) = c_{(1)} \otimes d_{(1)}\,\varepsilon_\Psi(c_{(3)} \otimes d_{(3)})\,\phi_\Psi(c_{(2)} \otimes d_{(2)})$$

$$= c_{(1)} \otimes d_{(1)}\,(\phi_\Psi * \varepsilon_\Psi)(c_{(2)} \otimes d_{(2)})\,.$$

This shows that ε_Ψ exists if and only if $(\phi_\Psi)^{-1}$ does, and in this case $\varepsilon_\Psi = (\phi_\Psi)^{-1}$. $\qquad\square$

As a corollary of this result we get a different proof of Lemma 4.3: suppose Ψ conormal, i.e. $(\phi_\Psi)^{-1} = \varepsilon_\otimes$, then $\phi_\Psi = \varepsilon_\otimes$ and hence $\Psi = \tau$.

Remark 4.7. In the above, the role of the two functionals ϕ_Ψ and ε_Ψ is completely symmetric: we have a second twisted coalgebra where ϕ^{-1} is used to deform the coproduct and ϕ is the compatible counit.

Starting with two twists associated to invertible functionals ϕ_1 and ϕ_2, it is natural to ask whether the composition $\Psi'_{\phi_1} \circ \Psi'_{\phi_2} = F^{-1}(\phi_1 \star \phi_2)$ is associated to an invertible functional. This is the case for the following class of coalgebras.

Definition 4.8. *A coalgebra C is said to be co-commutative up to isomorphism if there exists a coalgebra isomorphism $\chi : C \to C^{cop}$.*

If C is co-commutative up to isomorphism, then $C^{cop} \otimes D \simeq C \otimes D$ as coalgebras (with the tensor co-structures) and we have the induced algebra isomorphism $((C^{cop} \otimes D)', *) \simeq ((C \otimes D)', *)$.

Lemma 4.9. *Let C, D be coalgebras, and suppose at least one of them is co-commutative up to isomorphism. Then a functional $\phi : C \otimes D \to k$ is invertible with respect to the convolution product $*$ if and only if it is invertible with respect to the product $\star$.*

Proof. If C is co-commutative up to isomorphism, the result follows from the algebra isomorphisms

$$((C \otimes D)', \star) \simeq ((C^{cop} \otimes D)', *) \simeq ((C \otimes D)', *).$$

If D is co-commutative up to isomorphism, the result follows from the algebra isomorphisms

$$((C \otimes D)', \star) \simeq ((C^{cop} \otimes D)', *) \simeq ((C \otimes D^{cop})', *^{op}) \simeq ((C \otimes D)', *^{op}).$$

$\qquad\square$

Therefore whenever one among C and D is co-commutative up to isomorphism, if we compose two twists which admit counit (i.e. their associated functionals are convolution-invertible) then also the resulting twist admits counit.

From Thm. 4.5 it follows that a twist $\Psi' \in \mathsf{Tw}$ is invertible (and hence the associated twisted coproduct Δ_Ψ can be 'untwisted' with the inverse twist) if and only if ϕ_Ψ is invertible with respect the $\star$-product. On the other hand Δ_Ψ admits counit if and only if ϕ_Ψ is invertible with respect the convolution product. In general there may exist twists which do admit counit without being invertible and vice versa, see Example 4.13. [1]

---

[1] Instead if we consider twists which satisfy conditions (2.7)(2.9), the invertibility of the map Ψ implies the conormality. Indeed if Ψ satisfies (2.7), then $d^{[\Psi]} \otimes c^{[\Psi]} = \varepsilon((d_{(1)})^{[\Psi]})(d_{(2)})^{[[\Psi]]} \otimes (c^{[\Psi]})^{[[\Psi]]}$, where the notation $[[\Psi]]$ indicates a second application of Ψ. By applying to it $(id_C \otimes \varepsilon_D)\Psi^{-1}$ we get the right conormality $c\varepsilon_D(d) = c^{[\Psi]}\varepsilon_D(d^{[\Psi]})$. Similarly (2.9) implies the left conormality.

We return to the results of universality addressed in Sect. 3.1 for generic twists, now in the case of twists $\Psi' \in \mathsf{Tw}$. When the twisted counit exists, i.e. the functional corresponding to Ψ' is invertible, we have a second pair of projection maps q_C and q_D in addition to the projection maps p_C, p_D defined in (3.4).

Lemma 4.10. *Let $\Psi' \in \mathsf{Tw}$, with ϕ_Ψ invertible in $[(C \otimes D)', *]$. Then the maps $q_C : C \otimes_\Psi D \to C$ and $q_D : C \otimes_\Psi D \to D$ defined as*

$$q_C(c \otimes d) := \varepsilon_\Psi(c_{(1)} \otimes d) \, c_{(2)}, \qquad q_D(c \otimes d) := \varepsilon_\Psi(c \otimes d_{(2)}) \, d_{(1)}. \tag{4.7}$$

are coalgebra morphisms.

Proof. The map q_C trivially verifies $\varepsilon_C \circ q_C = \varepsilon_\Psi$. On generic elements $c \in C$, $d \in D$ we have

$$
\begin{aligned}
(q_C \otimes q_C) \circ \Delta_\Psi(c \otimes d) &= q_C(c_{(1)} \otimes d_{(1)}) \phi_\Psi(c_{(2)} \otimes d_{(2)}) \varepsilon_\Psi(c_{(3)} \otimes d_{(3)}) c_{(4)} \\
&= q_C(c_{(1)} \otimes d_{(1)})(\phi_\Psi * \varepsilon_\Psi)(c_{(2)} \otimes d_{(2)}) c_{(3)} \\
&= q_C(c_{(1)} \otimes d_{(1)}) \varepsilon_\otimes(c_{(2)} \otimes d_{(2)}) c_{(3)} \\
&= \Delta_C \circ q_C(c \otimes d)
\end{aligned}
$$

hence q_C is a coalgebra morphism. The proof for q_D is completely analogous. $\square$

Lemma 4.11. *Let $\Psi' \in \mathsf{Tw}$, with ϕ_Ψ invertible in $[(C \otimes D)', *]$. The maps $\nu := (p_C \otimes q_D)\Delta_\Psi : C \otimes_\Psi D \to C \otimes D$ and $\sigma := (q_C \otimes p_D)\Delta_\Psi : C \otimes_\Psi D \to C \otimes D$ are isomorphisms of vector spaces.*

Proof. By definition of the map ν, we have $\nu(c \otimes d) = c_{(1)} \otimes d_{(1)}(\phi_\Psi)^{-1}(c_{(2)} \otimes d_{(2)})$. It is invertible with inverse $\nu^{-1} : c \otimes d \mapsto c_{(1)} \otimes d_{(1)}\phi_\Psi(c_{(2)} \otimes d_{(2)})$. Analogously, $\sigma(c \otimes d) = (\phi_\Psi)^{-1}(c_{(1)} \otimes d_{(1)})c_{(2)} \otimes d_{(2)}$ and $\sigma^{-1} : c \otimes d \mapsto \phi_\Psi(c_{(1)} \otimes d_{(1)})c_{(2)} \otimes d_{(2)}$. $\square$

Corollary 4.12. *Let $\Psi' \in \mathsf{Tw}$ such that $\varepsilon_\Psi = \phi_\Psi^{-1}$ is invertible in $[(C \otimes D)', \star]$. Then $\widetilde\Psi = (p_D \otimes p_C)\Delta_\Psi \mu^{-1}$ is a conormal twist, different from τ, and we have the double isomorphisms $C \otimes_\Psi D \simeq C \otimes_{\widetilde\Psi} D \simeq C \otimes D$.*

Proof. The two different isomorphisms come from Corollary 3.7 applied to the twisted coalgebra $C \otimes_\Psi D$ and two different sets of projections. The first isomorphism $C \otimes_\Psi D \simeq C \otimes_{\widetilde\Psi} D$ is obtained by considering p_C and p_D as in (3.4). For the isomorphism $C \otimes_\Psi D \simeq C \otimes D$ we consider p_C in (3.4) and the projection q_D in (4.7). It is easy to show that in this case $\widetilde\Psi_\nu = (q_D \otimes p_C)\Delta_\Psi \nu^{-1} = \tau(p_C \otimes q_D)\Delta_\Psi \nu^{-1} = \tau$. (Alternatively, one could consider the pair q_C as in (4.7) and p_D in (3.4), with $\widetilde\Psi_\sigma = (p_D \otimes q_C)\Delta_\Psi \nu^{-1} = \tau$). $\square$

The previous result shows that all twist maps $\Psi' \in \mathsf{Tw}$ which admit compatible counit generate twisted coalgebras which are isomorphic to the untwisted one. At the same time we notice that in general $\widetilde\Psi \neq \tau$:

$$
\begin{aligned}
\widetilde\Psi(c \otimes d) &= (p_D \otimes p_C)\Delta_\Psi \mu^{-1}(c \otimes d) \\
&= (p_D \otimes p_C)\Delta_\Psi((\phi_\Psi^{-1})^\star(c_{(2)} \otimes d_{(1)}) \, c_{(1)} \otimes d_{(2)}) \\
&= (p_D \otimes p_C)\Big((\phi_\Psi^{-1})^\star(c_{(4)} \otimes d_{(1)}) \, \phi_\Psi(c_{(2)} \otimes d_{(3)}) \, c_{(1)} \otimes d_{(2)} \otimes c_{(3)} \otimes d_{(4)}\Big) \\
&= (\phi_\Psi^{-1})^\star(c_{(5)} \otimes d_{(1)}) \, \phi_\Psi(c_{(2)} \otimes d_{(4)}) \, \phi_\Psi^{-1}(c_{(1)} \otimes d_{(2)}) \, \phi_\Psi^{-1}(c_{(4)} \otimes d_{(5)}) \, d_{(3)} \otimes c_{(3)}.
\end{aligned}
$$

Nevertheless, these two different conormal twists give rise to isomorphic coalgebras. This leads to the natural problem to study a notion of equivalence for twists solutions of (2.2).

Example 4.13. As an example, we consider the case of C and D dually paired bialgebras. We take the pairing $\langle\,,\,\rangle : C \otimes D \to k$ as functional. The invertibility of the pairing, which determines the existence of the twisted counit, can be achieved by letting one among C and D to be a Hopf algebra. Let us require that C is a Hopf algebra (the case D Hopf algebra is completely equivalent). The convolution-inverse of $\langle\,,\,\rangle$ is $\langle\,,\,\rangle \circ (S_C \otimes \mathrm{id}_D)$ that we will denote with $\langle\,,\,\rangle^{-1}$.
The corresponding twisted coalgebra structure $(\Delta_{\langle,\rangle}, \varepsilon_{\langle,\rangle})$ on $C \otimes D$ is

$$\Delta_{\langle,\rangle}(c \otimes d) = \langle c_{(2)}\,,\,d_{(2)}\rangle\, c_{(1)} \otimes d_{(1)} \otimes c_{(3)} \otimes d_{(3)} \tag{4.8}$$

$$\varepsilon_{\langle,\rangle}(c \otimes d) = \langle S(c)\,,\,d\rangle \tag{4.9}$$

whereas the inverse of $\varepsilon_{\langle,\rangle}$ with respect to the $\star$ product exists and it is $\varepsilon_{\langle,\rangle}^{\star}(c \otimes d) = \langle S^2(c)\,,\,d\rangle$ as an easy computation shows.

As we already pointed out, whenever the twisted coproduct admits counit we also have the twisted structure associated to the convolution-inverse of the functional. This consists of $(C \otimes D, \Delta_{\langle,\rangle^{-1}}, \varepsilon_{\langle,\rangle^{-1}})$ with

$$\Delta_{\langle,\rangle^{-1}}(c \otimes d) = \langle S(c_{(2)})\,,\,d_{(2)}\rangle\, c_{(1)} \otimes d_{(1)} \otimes c_{(3)} \otimes d_{(3)} \tag{4.10}$$

$$\varepsilon_{\langle,\rangle^{-1}}(c \otimes d) = \langle c\,,\,d\rangle \tag{4.11}$$

We remark that in this case the inverse of $\varepsilon_{\langle,\rangle^{-1}}$ with respect to the $\star$ product exists if and only if the antipode S is invertible and in this case it is $\varepsilon_{\langle,\rangle^{-1}}^{\star}(c \otimes d) = \langle S^{-1}(c)\,,\,d\rangle$. This is an example of a twist in Tw which admits counit but that in general it is not invertible (see discussion after Lemma 4.9).

4.1. **An application.** In this section we consider C to be a bialgebra and D a coalgebra. (What follows can analogously be stated in the case C coalgebra and D bialgebra.)
Given any functional on $C \otimes D$ satisfying $\phi(c'c \otimes d) = \phi(c', d_{(1)})\phi(c, d_{(2)})$ for any $c, c' \in C$, $d \in D$, the map $\lambda_\phi : C \otimes D \to D$ defined as $\lambda_\phi(c \otimes d) := d_{(1)}\phi(c \otimes d_{(2)})$ is a left action of C on D. For example, for C and D dually paired bialgebras, the pairing satisfies the above condition and the associated map $\lambda_{\langle,\rangle}$ is the left coregular action of C on D.

One notable property of the twisted co-structures associated to a functional ϕ consists in the fact that they restore some kind of compatibility between the C action λ_ϕ and the coalgebra structure of D. The map λ_ϕ does *not* make D a left C-module coalgebra, unless $\phi = \varepsilon_\otimes$. Indeed D is a coalgebra in the category of left C-modules ${}_C M$ if the coproduct Δ_D and the counit ε_D are morphisms in ${}_C M$, and it is easy to check that this is not the case for $\phi \neq \varepsilon_\otimes$. An equivalent way to say that D is a left C-module coalgebra consists in requiring that λ_ϕ is a coalgebra map for $C \otimes D$ endowed with the tensor product co-structures $\Delta_\otimes$ and $\varepsilon_\otimes$. This amounts to the commutativity of the following

15

diagrams

$$
\begin{array}{ccc}
C \otimes D & \xrightarrow{\ \lambda_\phi\ } & D \\
{\scriptstyle \Delta_\otimes}\downarrow & & \downarrow{\scriptstyle \Delta_D} \\
C \otimes D \otimes C \otimes D & \xrightarrow{\ \lambda_\phi \otimes \lambda_\phi\ } & D \otimes D
\end{array}
\qquad\qquad
\begin{array}{ccc}
C \otimes D & \xrightarrow{\ \lambda_\phi\ } & D \\
& {\scriptstyle \varepsilon_\otimes}\searrow & \downarrow{\scriptstyle \varepsilon_D} \\
& & k
\end{array}
\qquad (4.12)
$$

which fail for $\phi \neq \varepsilon_\otimes$. However, as the next Proposition shows, we can make the above diagrams commutative by twisting in a suitable way the tensor co-structures $\Delta_\otimes$ and $\varepsilon_\otimes$. In this manner, after considering the twisted coalgebra co-structures on $C \otimes D$, the map λ_ϕ becomes a coalgebra map.

Proposition 4.14. *Let C be a bialgebra and D a coalgebra. Let $\lambda_\phi : C \otimes D \to D$ defined as $\lambda_\phi(c \otimes d) := d_{(1)}\phi(c \otimes d_{(2)})$ be a left action of C on D, with ϕ a convolution invertible functional on $C \otimes D$. Consider the twisted coalgebra $(C \otimes_{\phi^{-1}} D, \Delta_{\phi^{-1}}, \varepsilon_{\phi^{-1}})$. Then the following diagrams commute:*

$$
\begin{array}{ccc}
C \otimes D & \xrightarrow{\ \lambda_\phi\ } & D \\
{\scriptstyle \Delta_{\phi^{-1}}}\downarrow & & \downarrow{\scriptstyle \Delta_D} \\
C \otimes D \otimes C \otimes D & \xrightarrow{\ \lambda_\phi \otimes \lambda_\phi\ } & D \otimes D
\end{array}
\qquad\qquad
\begin{array}{ccc}
C \otimes D & \xrightarrow{\ \lambda_\phi\ } & D \\
& {\scriptstyle \varepsilon_{\phi^{-1}}}\searrow & \downarrow{\scriptstyle \varepsilon_D} \\
& & k
\end{array}
\qquad (4.13)
$$

so that λ_ϕ is a coalgebra map between $(C \otimes_{\phi^{-1}} D, \Delta_{\phi^{-1}}, \varepsilon_{\phi^{-1}})$ and $(D, \Delta_D, \varepsilon_D)$.

Proof. Let us verify the commutativity of the first diagram:

$$
\begin{aligned}
(\lambda_\phi \otimes \lambda_\phi)\Delta_{\phi^{-1}}(c \otimes d) &= \lambda_\phi(c_{(1)} \otimes d_{(1)})\phi^{-1}(c_{(2)} \otimes d_{(2)})\lambda_\phi(c_{(3)} \otimes d_{(3)}) \\
&= d_{(1)}\phi(c_{(1)} \otimes d_{(2)})\phi^{-1}(c_{(2)} \otimes d_{(3)})\lambda_\phi(c_{(3)} \otimes d_{(4)}) \\
&= d_{(1)}\lambda_\phi(c \otimes d_{(2)}) = \Delta_D(\lambda_\phi(c \otimes d)) \ .
\end{aligned}
$$

The commutativity of the diagram involving the counit is trivial. $\qquad\square$

4.2. **How to generate new twists from existing ones.** We have seen that the twists associated to functionals generate twisted coalgebras which eventually turn out to be isomorphic to the untwisted ones (see Corollary 4.12). Nevertheless, they can be composed with solutions of (2.3) to produce new twists.

Proposition 4.15. *Consider two twist maps χ and Ψ. Suppose that χ is a solution of (2.3) and $\Psi' \in \mathsf{Tw}$. Then the twist given by the composition $\chi_\Psi := \Psi' \circ \chi$ is a solution of (2.3).*

Proof. We compute both sides of eq. (2.3) for χ_Ψ by using the eqs. (4.1) and (4.2) to move Ψ to the left. The left hand-side is

$$
\begin{aligned}
(\mathrm{id}_D \otimes \mathrm{id}_C \otimes \Psi' \circ \chi)&(\mathrm{id}_D \otimes \Delta_C \otimes \mathrm{id}_D)(\Psi' \circ \chi \otimes \mathrm{id}_D)(\mathrm{id}_C \otimes \Delta_D) = \\
&= (\mathrm{id}_D \otimes \mathrm{id}_C \otimes \Psi' \circ \chi)(\Psi' \otimes \mathrm{id}_C \otimes \mathrm{id}_D)(\mathrm{id}_D \otimes \Delta_C \otimes \mathrm{id}_D)(\chi \otimes \mathrm{id}_D)(\mathrm{id}_C \otimes \Delta_D) = \\
&= (\Psi' \otimes \Psi') \,(\text{l.h.s of (2.3) for } \chi) \ .
\end{aligned}
$$

Similarly the right hand-side is

$$
\begin{aligned}
(\Psi' \circ \chi \otimes \mathrm{id}_D \otimes \mathrm{id}_C)&(\mathrm{id}_C \otimes \Delta_D \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \Psi' \circ \chi)(\Delta_C \otimes \mathrm{id}_D) = \\
&= (\Psi' \circ \chi \otimes \mathrm{id}_D \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \mathrm{id}_D \otimes \Psi')(\mathrm{id}_C \otimes \Delta_D \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \chi)(\Delta_C \otimes \mathrm{id}_D) \\
&= (\Psi' \otimes \Psi') \,(\text{r.h.s of (2.3) for } \chi) \ .
\end{aligned}
$$

16

$\square$

This result shows that the left composition by a $\Psi \in \mathsf{Tw}$ preserves the coassociativity. On the other side, we remark that the possible additional properties enjoyed by Ψ and χ are in general lost. For instance $\chi_\psi \notin \mathsf{Tw}$ for a generic $\chi \notin \mathsf{Tw}$. Furthermore if χ satisfies (2.7) and (2.9), the composition χ_Ψ in general is no longer a solution of these equations. In general if χ is conormal, the new twist χ_Ψ will not be conormal anymore (see Ex. 4.16) as can be seen with some algebra. Thus, provided the existence of a counit for χ_Ψ, the Prop. 4.15 above gives in particular a systematic way to obtain Z-conormal coassociative twists out of conormal ones.

At the present time, we lack of a general criterion for the existence of a compatible counit for χ_Ψ, thus its expression. Another interesting point to be investigated is the existence of projections to C and D, the study of the invertibility of their associated map μ (in view of Corollary 3.7) and the possible isomorphism between the twisted coalgebra $C \otimes_{\chi_\Psi} D$ and the starting one $C \otimes_\Psi D$.

Example 4.16. As an example of a new non-conormal twist χ_Ψ obtained from a conormal twist χ and which admits a compatible counit we consider the following. Let H be a bialgebra. Let C be a right H-module coalgebra with action $\triangleleft : C \otimes H \to C, c \otimes h \mapsto c \triangleleft h$. Let D be a right H-comodule coalgebra with coaction $\rho : D \to D \otimes H, d \mapsto d^{(0)} \otimes d^{(1)}$. The twist $\chi : c \otimes d \mapsto d^{(0)} \otimes c \triangleleft d^{(1)}$ is conormal and satisfies eqs. (2.7), (2.9) (cf. [3, Ex. 3.2]). The resulting twisted coalgebra $C \otimes_\chi D$ is Molnar's smash coproduct.
Let ϕ be a convolution-invertible functional on $C \otimes D$ such that $\phi(c \triangleleft h \otimes d) = \varepsilon_H(h) \phi(c \otimes d)$ for all $c \in C$, $d \in D$, $h \in H$. Then the composed twist $\chi_\phi := \chi_{\Psi_\phi} = \Psi'_\phi \circ \chi$ generates a coassociative coalgebra with counit $\varepsilon_{\chi_\phi} = \phi^{-1}$.

To prove this last statement we use the following identities following from the hypothesis: for all $c \in C$, $d \in D$, $h \in H$

$$(c \triangleleft h)_{(1)} \otimes (c \triangleleft h)_{(2)} = c_{(1)} \triangleleft h_{(1)} \otimes c_{(2)} \triangleleft h_{(2)} \quad ; \quad \varepsilon_C(c \triangleleft h) = \varepsilon_C(c)\varepsilon_H(h) \ ;$$

$$(d^{(0)})_{(1)} \otimes (d^{(0)})_{(2)} \otimes d^{(1)} = (d_{(1)})^{(0)} \otimes (d_{(2)})^{(0)} \otimes (d_{(1)})^{(1)}(d_{(2)})^{(1)} \quad ; \quad \varepsilon_D(d^{(0)})d^{(1)} = \varepsilon_D(d)1_H \ .$$

Firstly, we compute the twisted coproduct:

$$
\begin{aligned}
\Delta_{\chi_\phi}(c \otimes d) &= c_{(1)} \otimes \Psi'_\phi\left((d_{(1)})^{(0)} \otimes c_{(2)} \triangleleft (d_{(1)})^{(1)}\right) \otimes d_{(2)} \\
&= c_{(1)} \otimes \phi\left((c_{(2)} \triangleleft (d_{(1)})^{(1)})_{(1)} \otimes ((d_{(1)})^{(0)})_{(2)}\right)((d_{(1)})^{(0)})_{(1)} \otimes (c_{(2)} \triangleleft (d_{(1)})^{(1)})_{(2)} \otimes d_{(2)} = \\
&= c_{(1)} \otimes \phi\left(c_{(2)} \triangleleft ((d_{(1)})^{(1)})_{(1)} \otimes ((d_{(1)})^{(0)})_{(2)}\right)((d_{(1)})^{(0)})_{(1)} \otimes c_{(3)} \triangleleft ((d_{(1)})^{(1)})_{(2)} \otimes d_{(2)} = \\
&= c_{(1)} \otimes \phi\left(c_{(2)} \triangleleft \left((d_{(1)})^{(1)}(d_{(2)})^{(1)}\right)_{(1)} \otimes (d_{(2)})^{(0)}\right)(d_{(1)})^{(0)} \otimes c_{(3)} \triangleleft \left((d_{(1)})^{(1)}(d_{(2)})^{(1)}\right)_{(2)} \otimes d_{(3)} = \\
&= c_{(1)} \otimes \phi\left(c_{(2)} \otimes (d_{(2)})^{(0)}\right)(d_{(1)})^{(0)} \otimes c_{(3)} \triangleleft \left((d_{(1)})^{(1)}(d_{(2)})^{(1)}\right) \otimes d_{(3)}
\end{aligned}
$$

17

Next we show that $(\phi^{-1} \otimes \mathrm{id})\Delta_{\chi_\phi}(c \otimes d) = c \otimes d$:

$$
\begin{aligned}
(\phi^{-1} \otimes \mathrm{id})\Delta_{\chi_\phi}(c \otimes d) &= \phi^{-1}\left(c_{(1)} \otimes (d_{(1)})^{(0)}\right)\phi\left(c_{(2)} \otimes (d_{(2)})^{(0)}\right)c_{(3)} \triangleleft \left((d_{(1)})^{(1)}(d_{(2)})^{(1)}\right) \otimes d_{(3)} \\
&= \phi^{-1}\left(c_{(1)} \otimes ((d_{(1)})^{(0)})_{(1)}\right)\phi\left(c_{(2)} \otimes ((d_{(1)})^{(0)})_{(2)}\right)c_{(3)} \triangleleft (d_{(1)})^{(1)} \otimes d_{(2)} \\
&= c \triangleleft (d_{(1)})^{(1)}\varepsilon_H((d_{(1)})^{(0)}) \otimes d_{(2)} = c \triangleleft \varepsilon_D(d_{(1)}) \otimes d_{(2)} = c \otimes d
\end{aligned}
$$

and finally that $(\mathrm{id} \otimes \phi^{-1})\Delta_{\chi_\phi}(c \otimes d) = c \otimes d$:

$$
\begin{aligned}
(\mathrm{id} \otimes \phi^{-1})\Delta_{\chi_\phi}(c \otimes d) &= c_{(1)} \otimes \phi\left(c_{(2)} \otimes (d_{(2)})^{(0)}\right)(d_{(1)})^{(0)}\phi^{-1}\left(c_{(3)} \triangleleft ((d_{(1)})^{(1)}(d_{(2)})^{(1)}) \otimes d_{(3)}\right) \\
&= c_{(1)} \otimes \phi\left(c_{(2)} \otimes ((d_{(1)})^{(0)})_{(2)}\right)((d_{(1)})^{(0)})_{(1)}\phi^{-1}\left(c_{(3)} \triangleleft (d_{(1)})^{(1)} \otimes d_{(2)}\right) \\
&= c_{(1)} \otimes \phi\left(c_{(2)} \otimes d_{(2)}\right)d_{(1)}\phi^{-1}\left(c_{(3)} \otimes d_{(3)}\right) = c \otimes d
\end{aligned}
$$

where we have also made use of the property $\phi^{-1}(c \triangleleft h \otimes d) = \varepsilon_H(h)\phi^{-1}(c \otimes d)$ of ϕ^{-1}, inherited from the analogous property of ϕ.

5. On the equivalence of twists

Definition 5.1. *Two twists Ψ_1, Ψ_2 are said to be equivalent if there exists a coalgebra isomorphism $\theta : C \otimes D \to C \otimes D$ such that*

$$(\mathrm{id}_C \otimes \Psi_2' \otimes \mathrm{id}_D)(\theta \otimes \theta) = (\theta \otimes \theta)(\mathrm{id}_C \otimes \Psi_1' \otimes \mathrm{id}_D). \tag{5.1}$$

The above condition is for instance satisfied by any coalgebra isomorphism which factorizes as $\theta = \alpha \otimes \beta$ with $\alpha : C \to C$ and $\beta : D \to D$.

Lemma 5.2. *Suppose Ψ_1, Ψ_2 are two equivalent twists as in Def. 5.1. Then Ψ_1 is Z_1-conormal if and only if Ψ_2 is Z_2-conormal, with $Z_2 = \hat\theta\, Z_1\, \theta^{-1}$, where $\hat\theta := \tau\, \theta\, \tau$.*

Proof. Let us suppose that Ψ_1 is Z_1-conormal. Then

$$
\begin{aligned}
(\varepsilon_\otimes^\tau\, Z_2 \otimes \mathrm{id}_\otimes)\Delta_{\Psi_2} &= (\varepsilon_\otimes^\tau \otimes \mathrm{id}_\otimes)(\hat\theta Z_1 \theta^{-1} \otimes \mathrm{id}_\otimes)(\mathrm{id}_C \otimes \Psi_2' \otimes \mathrm{id}_D)\Delta_\otimes \\
&= (\varepsilon_\otimes^\tau \otimes \mathrm{id}_\otimes)(\hat\theta\, Z_1\, \theta^{-1} \otimes \mathrm{id}_\otimes)(\theta \otimes \theta)(\mathrm{id}_C \otimes \Psi_1' \otimes \mathrm{id}_D)\Delta_\otimes \theta^{-1} \\
&= (\varepsilon_\otimes^\tau \otimes \mathrm{id}_\otimes)(\hat\theta\, Z_1 \otimes \theta)\Delta_{\Psi_1}\theta^{-1} \\
&= \theta(\varepsilon_\otimes^\tau Z_1 \otimes \mathrm{id}_\otimes)\Delta_{\Psi_1}\theta^{-1} = \mathrm{id}_\otimes
\end{aligned}
$$

and similarly $(\mathrm{id}_\otimes \otimes \varepsilon_\otimes^\tau\, Z_2)\Delta_{\Psi_2} = \mathrm{id}_\otimes$. The opposite implication is analogous. $\square$

We observe that hence in particular a conormal twist can only be equivalent to another conormal twist. Equivalent twists generate isomorphic twisted coalgebras:

Proposition 5.3. *Let Ψ_1, Ψ_2 be two equivalent twists as in Def. 5.1 (which satisfy (2.3)). Then the map $\theta : (C \otimes_{\Psi_1} D, \Delta_{\Psi_1}) \to (C \otimes_{\Psi_2} D, \Delta_{\Psi_2})$ intertwines the (coassociative) coproducts Δ_{Ψ_1} and Δ_{Ψ_2}. In particular if Δ_{Ψ_1}, Δ_{Ψ_2} admit counits, then the map θ is an isomorphism of coalgebras.*

Proof. From (5.1), it is straightforward that θ satisfies $(\theta \otimes \theta)\Delta_{\Psi_1} = \Delta_{\Psi_2}\theta$. From Lemma 5.2 it follows easily that $\varepsilon_{\Psi_1} = \varepsilon_{\Psi_2}\theta$. $\square$

We notice that this notion of equivalence of twists is not exhaustive to capture when two twisted coalgebras are isomorphic. For instance in Corollary 4.12 we have two coalgebra isomorphisms $C \otimes_\Psi D \simeq C \otimes D$ and $C \otimes_\Psi D \simeq C \otimes_{\widetilde{\Psi}} D$, but Ψ is not conormal while $\widetilde{\Psi}$ does. Furthermore $C \otimes D \simeq C \otimes_{\widetilde{\Psi}} D$ but from Def. 5.1, the equivalence class of the trivial twist consists of the sole τ.

Equivalent twists characterize a finer notion of isomorphism of twisted coalgebras.

Definition 5.4. *Two twisted coalgebras* $(C \otimes_{\Psi_1} D, \Delta_{\Psi_1})$, $(C \otimes_{\Psi_2} D, \Delta_{\Psi_2})$ *are said to be strongly isomorphic if there exists a coalgebra isomorphism* $\theta : (C \otimes_{\Psi_1} D, \Delta_{\Psi_1}) \to (C \otimes_{\Psi_2} D, \Delta_{\Psi_2})$ *which in addition intertwines the twist maps in the sense that*

$$(\mathrm{id}_C \otimes \Psi_2' \otimes \mathrm{id}_D)(\theta \otimes \theta) = (\theta \otimes \theta)(\mathrm{id}_C \otimes \Psi_1' \otimes \mathrm{id}_D) .$$

From Prop. 5.3 we immediately get

Proposition 5.5. *Two twisted coalgebras* $(C \otimes_{\Psi_1} D, \Delta_{\Psi_1})$, $(C \otimes_{\Psi_2} D, \Delta_{\Psi_2})$ *are strongly isomorphic if and only if the twists* Ψ_1, Ψ_2 *are equivalent.*

6. The dual case: twisting tensor algebra structures

Notation. Given a unital k-algebra A, we denote by m_A and η_A the multiplication and the unit map respectively. We also use the standard notation $1_A = \eta_A(1_k)$ for the unit element of A. We denote with A^{op} the opposite algebra $(A, m_A \circ \tau, \eta_A)$. Given another algebra B, when we refer to $A \otimes B$ as an algebra, without further specifications, we always mean $(A \otimes B, m_\otimes, \eta_\otimes)$ with the standard tensor product structures $m_\otimes = (m_A \otimes m_B) \circ (id \otimes \tau \otimes id)$, $\eta_\otimes = \eta_A \otimes \eta_B$.

In this section we dualize the results obtained in the study of twisted tensor co-structures to the dual case of the tensor product of two algebras. We will omit to write most of the proofs. We consider two (fixed) unital and associative k-algebras A and B. We refer to a k-linear map $\psi : B \otimes A \to A \otimes B$, $\psi(b \otimes a) := a^{[\psi]} \otimes b^{[\psi]}$ to as a 'twist map' and use the same terminology for the map $\psi' := \tau \circ \psi$. We recall some relevant results from [3, §2].

Theorem 6.1. *Let* ψ *be a twist map. The map* $m_\psi : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$ *given by*

$$m_\psi := (m_A \otimes m_B)(\mathrm{id}_A \otimes \psi \otimes \mathrm{id}_B) \tag{6.1}$$

defines an associative product on the vector space $A \otimes B$ *if and only if the twist map* ψ *satisfies*

$$(\mathrm{id}_A \otimes m_B)(\psi \otimes \mathrm{id}_B)(\mathrm{id}_B \otimes m_A \otimes \mathrm{id}_B)(\mathrm{id}_B \otimes \mathrm{id}_A \otimes \psi) =$$
$$(m_A \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \psi)(\mathrm{id}_A \otimes m_B \otimes \mathrm{id}_A)(\psi \otimes \mathrm{id}_B \otimes \mathrm{id}_A) . \tag{6.2}$$

We denote with $A \otimes_\psi B$ the vector space $A \otimes B$ equipped with the twisted product m_ψ. It is possible to make $A \otimes_\psi B$ into a unital algebra provided that ψ satisfies some further conditions. The most common choice is the following:

Definition 6.2. *A twist map* ψ *is said to be right (resp. left) normal if for all* $a \in A$, $b \in B$

$$\psi(1_B \otimes a) = a \otimes 1_B , \quad (resp. \ \psi(b \otimes 1_A) = 1_A \otimes b) . \tag{6.3}$$

It is said to be normal when it is both right and left normal.

Lemma 6.3. *Let ψ be a twist map. The tensor unit $\eta_\otimes := \eta_A \otimes \eta_B$ is compatible with the twisted product m_ψ, i.e. $m_\psi \circ (\mathrm{id} \otimes \eta_\otimes) = \mathrm{id} = m_\psi \circ (\eta_\otimes \otimes \mathrm{id})$ if and only if ψ is normal.*

The associativity of m_ψ and the property of $\eta_\otimes$ to be a compatible unit are completely independent and clearly there are examples of associative algebras $(A \otimes B, m_\psi)$ which do not admit $\eta_\otimes$ as unit (or even more in general which do not admit unit at all) and there are normal twists ψ which do not satisfy the associativity condition (6.2).

Under the assumption of normality of ψ, the condition (6.2) can be split in the following two conditions

$$(\mathrm{id}_A \otimes m_B)(\psi \otimes \mathrm{id}_B)(\mathrm{id}_B \otimes \psi) = \psi(m_B \otimes \mathrm{id}_A) \tag{6.4}$$

and

$$(m_A \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \psi)(\psi \otimes \mathrm{id}_A) = \psi(\mathrm{id}_B \otimes m_A). \tag{6.5}$$

Theorem 6.4. *Let A, B and X be unital k-algebras. The following two conditions are equivalent:*
  (1) *There exists an algebra isomorphism $X \simeq A \otimes_\psi B$ for some normal twist map ψ solution of (6.2);*
  (2) *X factorizes through A and B, i.e. there exist algebra morphisms $u_A : A \to X$ and $u_B : B \to X$ such that the map $\xi := m_X \circ (u_A \otimes u_B) : A \otimes B \to X$ is an isomorphism of vector spaces.*

The proof is based on the fact that if $A \otimes_\psi B$ is an algebra associated to a normal twist ψ, then the maps

$$\begin{aligned} \pi_A : A \to A \otimes_\psi B, \quad & a \mapsto a \otimes 1_B \\ \pi_B : B \to A \otimes_\psi B, \quad & b \mapsto 1_A \otimes b \end{aligned} \tag{6.6}$$

are algebra morphisms, and the map

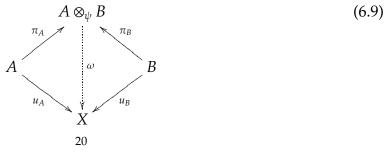$$\xi := m_\psi \circ (\pi_A \otimes \pi_B) : A \otimes B \to A \otimes_\psi B \tag{6.7}$$

is an isomorphism of vector spaces, being $\xi(a \otimes b) = a \otimes b$. For the opposite implication, one constructs the normal twist map as $\psi = \xi^{-1} \circ m_X \circ (u_B \otimes u_A)$.

Twisted algebras $A \otimes_\psi B$ are characterized by the following universal property:

Proposition 6.5. *Let A and B be algebras, ψ a normal twist satisfying (6.2). Let X be a unital algebra and $u_A : A \to X$, $u_B : B \to X$ algebra morphisms such that*

$$m_X \circ (u_B \otimes u_A) = m_X \circ (u_A \otimes u_B) \circ \psi. \tag{6.8}$$

Then there exists a unique algebra morphism $\omega : A \otimes_\psi B \to X$ such that the following diagram commutes

$$\tag{6.9}$$

20

6.1. **Non-normal twists.** We address to twist maps ψ which are not normal but still solutions of the associativity constraint (6.2). We generalize the notion of normality into the following

Definition 6.6. *A twist map ψ is said to be right z-normal if there exists an element $z = z_A \otimes z_B \in A \otimes B$ (possible sum understood) such that*

$$z_A a^{[\psi]} \otimes (z_B)^{[\psi]} = a \otimes 1_B, \quad \forall a \in A \tag{6.10}$$

and it is said to be left z-normal if

$$(z_A)^{[\psi]} \otimes b^{[\psi]} z_B = 1_A \otimes b, \quad \forall b \in B. \tag{6.11}$$

It is said to be Z-normal if it is both left and right z-normal.

Lemma 6.7. *Let ψ be a twist map.*

(i) *ψ is left (resp. right) normal if and only if ψ is left (resp. right) $1_A \otimes 1_B$-normal.*
(ii) *If ψ is left (resp. right) normal, then ψ is left (resp. right) $\psi(1_B \otimes 1_A)$-normal.*
(iii) *If ψ satisfies conditions (6.4)(6.5), then ψ is left (resp. right) $\psi(1_B \otimes 1_A)$-normal if and only if ψ is left (resp. right) normal.*

Proposition 6.8. *A twist map ψ which satisfies (6.2) is z-normal for some $z = z_A \otimes z_B \in A \otimes B$ if and only if the k-linear map*

$$\eta_z = k \to A \otimes B, \ 1 \mapsto z_A \otimes z_B \tag{6.12}$$

is a compatible unit for $A \otimes_\psi B$.

In the following, we will rather use the notation η_ψ to indicate the unit η_z of the algebra $A \otimes_\psi B$ associated to a z-normal twist ψ.

6.2. **Universal properties and factorization.** In this subsection we consider a generic twist map ψ such that the (non necessarily associative) algebra $(A \otimes_\psi B, m_\psi)$ is unital, with unit η_ψ, $\eta_\psi(1) = z_A \otimes z_B$. Further properties, for example the associativity condition (6.2), are explicitly required only when necessary.

Lemma 6.9. *The inclusion maps $i_A : A \to A \otimes_\psi B$, $i_B : B \to A \otimes_\psi B$ defined respectively by*

$$i_A(a) := a z_A \otimes z_B, \quad i_B(b) := z_A \otimes z_B b, \quad \forall a \in A, \ b \in B. \tag{6.13}$$

are algebra morphisms.

Proof. (sketch). Using the notation $\cdot$ to indicate the untwisted tensor product multiplication and $\cdot_\psi$ for the twisted multiplication, one can observe that for all $a, a' \in A$ and $b, b' \in B$ it is

$$[(a \otimes 1_B) \cdot (z_A \otimes z_B)] \cdot_\psi (a' \otimes b) = (a \otimes 1_B) \cdot [(z_A \otimes z_B) \cdot_\psi (a' \otimes b)]$$

and

$$(a \otimes b) \cdot_\psi [(z_A \otimes z_B) \cdot (1 \otimes b')] = [(a \otimes b) \cdot_\psi (z_A \otimes z_B)] \cdot (1 \otimes b')$$

and hence prove that i_A and i_B are algebra morphisms. $\square$

Clearly, these maps reduce to the inclusions π_A, π_B in (6.6) when the twist ψ is normal.

21

Proposition 6.10. *Let ψ be a twist map and $z_a \otimes z_B$ the unit element of $A \otimes_\psi B$. Let i_A, i_B be the algebra maps defined in* (6.13). *The map $\mu : A \otimes B \to A \otimes_\psi B$,*

$$\mu := m_\psi(i_A \otimes i_B) : \quad a \otimes b \mapsto a z_A \otimes z_B b$$

is an isomorphism of vector spaces if and only if the unit element $z_A \otimes z_B$ is invertible in the algebra $A \otimes B^{op}$.

Proof. (sketch). Let us first assume that $z_A \otimes z_B$ is invertible in $A \otimes B^{op}$ and denote its inverse by $z_A^\star \otimes z_B^\star$, so that $z_A z_A^\star \otimes z_B^\star z_B = 1_A \otimes 1_B = z_A^\star z_A \otimes z_B z_B^\star$. Then

$$\mu^{-1}(a \otimes b) = a z_A^\star \otimes z_B^\star b$$

is the inverse of μ. Conversely, if μ is invertible, then $z_A^\star \otimes z_B^\star := \mu^{-1}(1_A \otimes 1_B)$ is the inverse of the unit element $z_A \otimes z_B$ in $A \otimes B^{op}$. $\qquad \square$

Note that the above result holds for a generic twist, not necessarily solutions of (6.2).
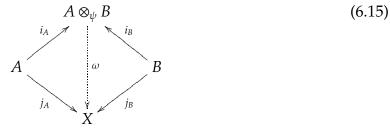
As a consequence of Thm. 6.4 we have the following

Corollary 6.11. *Let $(A \otimes_\psi B, m_\psi, \eta_\psi)$ be a twisted algebra associated to a twist map ψ which satisfies the associativity condition* (6.2). *Suppose there exist two algebra morphisms $i_A : A \to A \otimes_\psi B$, $i_B : B \to A \otimes_\psi B$ such that the map $\mu = m_\psi(i_A \otimes i_B)$ is invertible. Then $\widetilde{\psi} := \mu^{-1} m_\psi(i_B \otimes i_A)$ is a normal twist, and $\mu : A \otimes_\psi B \to A \otimes_{\widetilde{\psi}} B$ realizes an algebra isomorphism.*

We conclude with the following universal characterization of the twisted algebra $A \otimes_\psi B$ (cf. Prop. 6.5):

Theorem 6.12. *Let $A \otimes_\psi B = (A \otimes B, m_\psi, \eta_\psi)$ be a twisted algebra for some twist ψ solution of* (6.2), *and i_A, i_B the inclusions introduced in* (6.13). *Let X be an associative algebra, together with algebra morphisms $j_A : A \to X$, $j_B : B \to X$ such that*

$$m_X \circ (j_B \otimes j_A) = m_X \circ (j_A \otimes j_B) \circ \psi, \qquad m_X \circ (j_A \otimes j_B) \circ \eta_\psi = \eta_X. \qquad (6.14)$$

Then there exists an algebra morphism $\omega : A \otimes_\psi B \to X$ such that $j_A = \omega \circ i_A$ and $j_B = \omega \circ i_B$, i.e. the following diagram commutes

$$(6.15)$$

Proof. (sketch). Set $\omega := m_X \circ (j_A \otimes j_B)$. $\qquad \square$

Remark 6.13. In the above Theorem the algebra map ω is unique for twists ψ such that the corresponding map $\mu = m_\psi \circ (i_A \otimes i_B)$ is surjective.

Remark 6.14. Suppose X itself is a unital twisted algebra $X = A \otimes_\chi B$ associated to some twist χ solution of (6.2), and the map $\mu_\chi = m_\chi \circ (i_A \otimes i_B)$ is invertible (i.e. the case of Corollary 6.11). Then condition (6.14) is satisfied by the normal twist $\psi = \widetilde{\chi} = \mu_\chi^{-1} \circ m_\chi(i_B \otimes i_A)$ and the algebra isomorphism $\omega : A \otimes_\chi B \to A \otimes_{\widetilde{\chi}} B$ is the same of Corollary 6.11.

6.3. Twists from morphisms.

In analogy with what done in Sect. 4, we now describe a class of solutions of condition (6.2) which are intrinsically non normal. Conditions (6.4)(6.5) are no longer necessary for the associativity of m_ψ once we remove the condition of normality.

We denote with $_BM_A$ the category of left B-modules and right A-modules. We have that $B \otimes A \in Obj(_BM_A)$ via the left B-action $_B\rho$ and the right A-action ρ_A defined on generic $a, a' \in A$ and $b, b' \in B$ as

$$_B\rho(b' \otimes b \otimes a) = b'b \otimes a \qquad \rho_A(b \otimes a \otimes a') = b \otimes aa' \,.$$

Similarly, we have that $A \otimes B \in Obj(_BM_A)$ via the left B-action $_B\rho^\tau$ and the right A-action ρ_A^τ defined as

$$_B\rho^\tau(b' \otimes a \otimes b) = a \otimes b'b \qquad \rho_A^\tau(a \otimes b \otimes a') = aa' \otimes b \,.$$

Theorem 6.15. *Any twist map ψ which is a morphism in $_BM_A$ satisfies condition (6.2) and hence generates an associative product m_ψ on $A \otimes B$.*

Proof. (sketch). We use $R \in Mor(_BM_A)$ to move R towards right in both sides of (6.2), so that it becomes

$$(m_A \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes {_B\rho^\tau})(R \otimes R) = (\mathrm{id}_A \otimes m_B)(\rho_A^\tau \otimes \mathrm{id}_B)(R \otimes R) \,.$$

It is a direct computation to check that $(m_A \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes {_B\rho^\tau}) = (\mathrm{id}_A \otimes m_B)(\rho_A^\tau \otimes \mathrm{id}_B)$ on $A \otimes B \otimes A \otimes B$. $\qquad\square$

We observe that $\psi \in Mor(_BM_A)$ if and only if the map $\psi' = \tau \circ \psi : B \otimes A \to B \otimes A$ does. The condition $\psi' \in Mor(_BM_A)$ can be expressed as the validity of the conditions

$$\psi'(m_B \otimes \mathrm{id}_A) = (m_B \otimes \mathrm{id}_A)(\mathrm{id}_B \otimes \psi') \tag{6.16}$$

$$\psi'(\mathrm{id}_B \otimes m_A) = (\mathrm{id}_B \otimes m_A)(\psi' \otimes \mathrm{id}_A) \tag{6.17}$$

This allows to conclude that

Corollary 6.16. *The space of solutions of (6.16) and (6.17) is a unital algebra with multiplication given by the composition of morphisms.*

Furthermore, the above conditions (6.16) and (6.17) are equivalent to the following ones

$$b^{[\psi]} \otimes a^{[\psi]} = b(1_B)^{[\psi]} \otimes a^{[\psi]} \tag{6.18}$$

$$b^{[\psi]} \otimes a^{[\psi]} = b^{[\psi]} \otimes (1_A)^{[\psi]}a \tag{6.19}$$

for all $a \in A$, $b \in B$ and these yield a simplified expression of the twisted product:

$$m_\psi(a \otimes b \otimes a' \otimes b') = aa'1_A{}^{[\psi]} \otimes 1_B{}^{[\psi]}bb' \,, \quad \forall a, a' \in A, \ \forall b, b' \in B \,. \tag{6.20}$$

Let tw be the set of endomorphisms of $B \otimes A$ which are morphisms in $_BM_A$; tw is a unital algebra with the composition of morphisms in $_BM_A$.

Theorem 6.17. *There is an algebra isomorphism $G : \mathsf{tw} \to A \otimes B^{op}$.*

Proof. (sketch). Given $\psi' \in$ tw we define $G(\psi') := \psi(1_B \otimes 1_A)$. Conversely, fixed an element $\bar{a} \otimes \bar{b} \in A \otimes B$, we define the inverse map G^{-1} to be

$$[G^{-1}(\bar{a} \otimes \bar{b})](a \otimes b) := \bar{a}a \otimes b\bar{b}.$$

□

Proposition 6.18. *Given a twist map $\psi' \in$ Tw, the associated twisted algebra $A \otimes_\psi B = (A \otimes B, m_\psi)$ is unital if and only if $\psi(1_B \otimes 1_A)$ is invertible in $A \otimes B$, and in this case the unit map η_ψ is defined by $\eta_\psi(1) = (1_A{}^{[\psi]} \otimes 1_B{}^{[\psi]})^{-1}$.*

We call an algebra *commutative up to isomorphism* if it is isomorphic to its opposite algebra. Whenever one among A and B is commutative up to isomorphism, if we consider two twists whose resulting twisted tensor algebras are unital, then also the algebra corresponding to the composition of the two twists admits unit:

Lemma 6.19. *Let A, B be algebras, and suppose at least one of them is commutative up to isomorphism (say $B \simeq B^{op}$). Then an element $a \otimes b \in A \otimes B$ is invertible in $A \otimes B$ if and only if it is invertible as an element in $A \otimes B^{op}$.*

Summarizing, a twist $\psi' \in$ Tw is invertible (and hence the associated twisted product m_ψ can be 'untwisted' with the inverse twist) if and only if $\psi(1_B \otimes 1_A)$ is invertible in $A \otimes B^{op}$. On the other hand $(A \otimes B, m_\psi)$ admits unit if and only if $\psi(1_B \otimes 1_A)$ is invertible in $A \otimes B$.

A last observation is that if ψ is not invertible, in particular not injective and $\bar{a} \otimes \bar{b} \in ker(\psi)$, then the twisted algebra $A \otimes_\psi B$ has zero divisors: $(a \otimes \bar{b}) \cdot_\psi (\bar{a} \otimes b) = 0$, for all $a \in A$, $b \in B$.

We return to the results of universality addressed in Sect. 6.2 for generic twists, now in the case of twists $\psi' \in$ Tw. When the twisted algebra $A \otimes_\psi b$ is unital, we have a second pair of inclusion maps h_A and h_B in addition to the inclusion maps i_A, i_B defined in (6.13).

Lemma 6.20. *Let $\psi' \in$ Tw such that $A \otimes_\psi B$ is unital. Let $z_A \otimes z_B$ be the unit element. Then the maps $h_A : A \to A \otimes_\psi B$ and $h_B : A \to A \otimes_\psi B$ defined as*

$$h_A(a) := z_A a \otimes z_B, \qquad h_B(b) := z_A \otimes b z_B, \quad a \in A, \ b \in B \tag{6.21}$$

are algebra morphisms.

Lemma 6.21. *Let $\psi' \in$ Tw such that $A \otimes_\psi B$ is unital with unit element $z_A \otimes z_B$. The maps $\nu := m_\psi(i_A \otimes h_B) : A \otimes B \to A \otimes_\psi B$ and $\sigma := m_\psi(h_A \otimes i_B) : A \otimes B \to A \otimes_\psi B$ are isomorphisms of vector spaces.*

Proof. (sketch). Let $a \in A$, $b \in B$. By definition of the map ν, we have $\nu(a \otimes b) = a z_A \otimes b z_B$. It is invertible with inverse $\nu^{-1}(a \otimes b) = a(1_A)^{[\psi]} \otimes b(1_B)^{[\psi]}$. Analogously, $\sigma(a \otimes b) = z_a a \otimes z_B b$ and $\sigma^{-1}(a \otimes b) = (1_A)^{[\psi]} a \otimes (1_B)^{[\psi]} b$. □

Corollary 6.22. *Let $\psi' \in$ Tw such that such that $A \otimes_\psi B$ is unital and the unit element is invertible in $A \otimes B^{op}$. Then $\widetilde{\psi} = \mu^{-1} m_\psi(i_B \otimes i_A)$ is a normal twist, different from τ, and we have the double isomorphisms $A \otimes_\psi B \simeq A \otimes_{\widetilde{\psi}} B \simeq A \otimes B$.*

We conclude by stressing that the twists $\psi' \in \mathsf{tw}$ can be composed with solutions of (6.2) to produce new twists (cf. Sect. 4.2):

Proposition 6.23. *Consider two twist maps χ and ψ. Suppose that χ is a solution of* (6.2) *and $\psi' \in \mathsf{Tw}$. Then the twist given by the composition $\chi_\Psi := \chi \circ \psi'$ is a solution of* (6.2)*.*

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